<u>Lecture 20</u> <u>Homogeneous</u> <u>Second</u>-<u>Order</u> <u>Linear</u> <u>ODE</u>!

We're in the homestretch!

<u>This week</u>

<u>Wed</u> No Office Hours

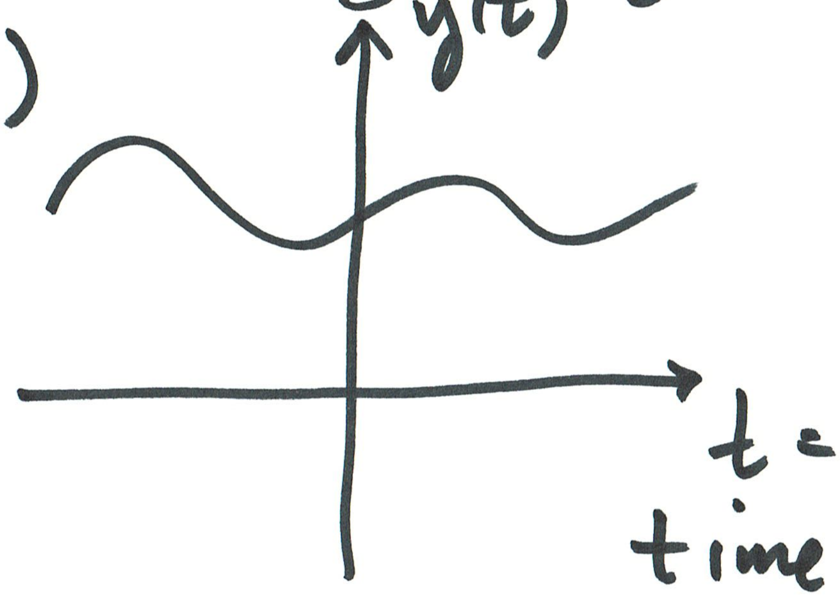
<u>Thurs</u> Guest lecturer... Be kind!

<u>Fri</u> No section, no quiz

"I became an atheist because, as a graduate student studying quantum physics, life seemed to be reducible to second-order differential equations." Francis Collins
NIH Director

Warmup 1) Find **all** solns to $y' = ay$

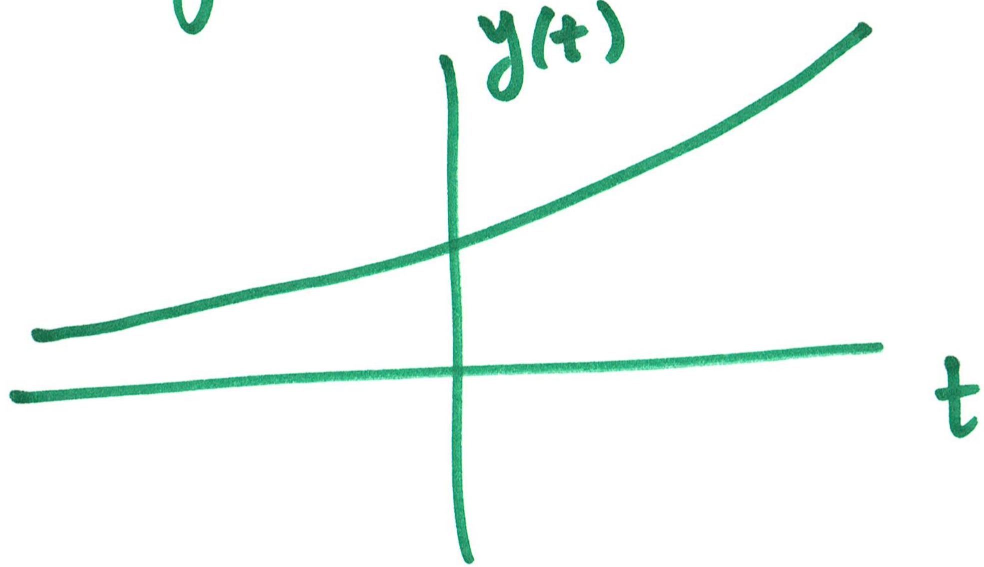
$y = y(t)$



2) Find (**unique**) soln to $y' = ay$ such that $y(0) = 3$.

IVP = Initial Value Problem

__Soln__ 1) $y = ce^{at}$, c any number



__Claim__ $y = ce^{at}$ are __all__ possible solns

__why?__ Suppose f is a soln

$$f' = af$$

__Want__: $f = ce^{at}$ for some c.

__Introduce__ $g = fe^{-at}$ __Expect__ g is const.

$$g' = f'e^{-at} + f\cdot(-a)e^{-at}$$
$$= afe^{-at} + (-a)fe^{-at} = 0!$$

__So__ $g = c$ some const.

__Conclude__ $f = ce^{at}$

2) Now <u>IVP</u>: find soln with $y(0) = 3$.

$$y(0) = ce^{a \cdot 0} = c$$

<u>Set</u> $c = 3$. <u>Soln to</u> <u>IVP</u>: $y = 3e^{at}$.

A word about our fave fn $y = e^{at}$

1) $y = e^{at}$ solves $y' = ay$

and any soln is a scale of it

2) $e^{(a+b)t} = e^{at} \cdot e^{bt}$

2') $e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t}$

$\overset{\text{Euler}}{=} e^{\alpha t}\left(\cos(\beta t) + i \sin(\beta t)\right)$

3) $e^{at} = 1 + \frac{(at)}{1!} + \frac{(at)^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{(at)^n}{n!}$

Lin Alg Interpretation V = vect sp of ∞-diff. fns $f : \mathbb{R} \to \mathbb{R}$

$T : V \to V$ lin transf

$T = \frac{d}{dt}$ so $T(f) = \frac{d}{dt}(f) = f'$

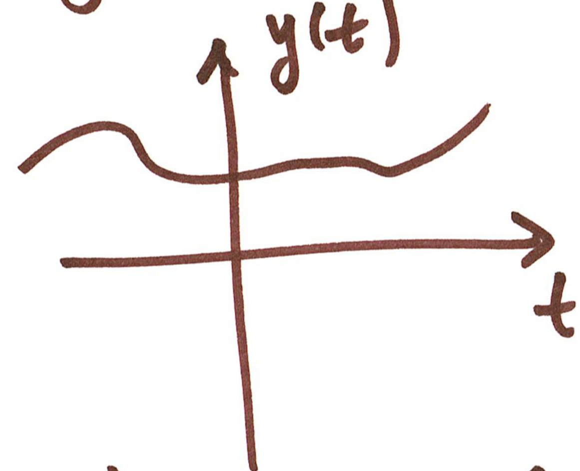
Solve $T(f) = af$ e-vector eqn for e-value a!

Gen solns form e-space $E_a := \text{Span}\{e^{at}\}$

IVP $S : E_a \to \mathbb{R}$
$S(f) = f(0)$

solns solve lin sys $S(f) = 3$

Now onto 2nd order ODE $y = y(t)$



$$y'' + by' + cy = 0$$

numbers

zero: homogeneous

also IVP $y(0) = Y_0$

$y'(0) = Y_1$

numbers.

<u>Lin</u> Alg <u>Interpretation</u> V = vect sp of ∞-diff fns $f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$-valued)

$T: V \to V$

$T = \left(\frac{d^2}{dt^2} + b\frac{d}{dt} + c \right)$

<u>Solve</u> eqn: $T(y) = 0$

Find $\mathrm{Null}(T)$!

<u>IVP</u> eqn $S: \mathrm{Null}(T) \to \mathbb{R}^2$

$S(y) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$

Solve $S(y) = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$!

Caution:

$$\frac{d^2}{dt^2} y \neq \left(\frac{d}{dt} y\right)^2$$

Useful Def auxilliary eq$\underline{n}$ $\quad r = \frac{d}{dt}$

$$r^2 + br + c = 0$$

Qud eq$\underline{n}$ gives roots r_1, r_2

Cases

1) $r_1 \neq r_2$ distinct real roots
2) $r_1 = r_2$ repeated real root
3) $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ complex conjugate roots

<u>Exer</u> 1) Find <u>all</u> solns of

$$y'' - 3y' - 4y = 0$$

2) Solve <u>IVP</u>

$$y(0) = 0 \ , \ y'(0) = 1$$

<u>Soln</u> Aux eqn $r^2 - 3r - 4 = 0$

roots $r_1 = 4, r_2 = -1$

eqn <u>factors</u> $(r-4)(r-(-1)) = 0$

distinct real roots (Case (1))

Key idea: original ODE also factors!

$$\left(\frac{d}{dt} - 4\right)\left(\frac{d}{dt} - (-1)\right) y = 0$$

(order is unimportant)

Solns of $\left(\frac{d}{dt} - (-1)\right) y = 0$

will also solve $y'' - 3y' - 4y = 0$!

Similarly for $\left(\frac{d}{dt} - 4\right) y = 0$

<u>Find 2 dims</u> of <u>solns</u>

<u>roots</u>

<u>$r_1 = 4$</u> $\left(\frac{d}{dt} - 4\right) y = 0$ $y_1 = e^{r_1 t} = e^{4t}$

<u>$r_2 = -1$</u> $\left(\frac{d}{dt} - (-1)\right) y = 0$ $y_2 = e^{r_2 t} = e^{-1 \cdot t}$

<u>Fact</u> Any soln is of form

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{4t} + c_2 e^{-t}$$

In other words

Solns of $y'' - 3y' - 4y = 0$

form Span$\{y_1 = e^{4t}, y_2 = e^{-t}\}$

<u>2</u> dim <u>vect</u> sp.

Seeing is believing: Take $y_* = e^{4t}$

Plug in to $y'' - 3y' - 4y = 0$

$$16e^{4t} - 12e^{4t} - 4e^{4t} = 0$$

~~Magic!~~

Now solve IVP $\quad y = c_1 y_1 + c_2 y_2$

Solve for c_1, c_2 so that $y(0) = 0$

$y'(0) = 1$

Evaluate

$$y(0) = c_1 y_1(0) + c_2 y_2(0)$$

$$= c_1 + c_2 \quad = 0$$

$$y'(0) = c_1 y_1'(0) + c_2 y_2'(0)$$

$$= 4c_1 + (-1)c_2 \quad = 1$$

This is a lin syst!

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & -1 & 1 \end{array}\right]$$

$\ldots \quad c_1 = \frac{1}{5}, \quad c_2 = -\frac{1}{5}$

Soln to <u>IVP</u> $\quad y = \frac{1}{5}e^{4t} + \left(-\frac{1}{5}\right)e^{-t}$

__Exer__ 1) Find __all__ solns to

$$y'' + 6y' + 9y = 0$$

2) Solve IVP with

$$y(0) = 1, \quad y'(0) = 0$$

__Soln__ aux eqn $r^2 + 6r + 9 = 0$

factors $(r+3)(r+3) = 0$

repeated real root (Case (2))

Factor original ODE

$$\left(\frac{d}{dt}+3\right)\left(\frac{d}{dt}+3\right)y = 0$$

We find soln $y_1 = e^{-3t}$

Where is expected second soln?

Try $y_2 = te^{-3t}$

<u>Check</u> $\left(\frac{d}{dt}+3\right)\left(\frac{d}{dt}+3\right)te^{-3t}$

$$= \left(\frac{d}{dt}+3\right)\left(e^{-3t}\right) = 0$$

General soln $y = c_1 e^{-3t} + c_2 t e^{-3t}$

Solve IVP:

$$y(0) = c_1 = 1$$

$$y'(0) = -3c_1 + c_2 = 0$$

Lin sys $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -3 & 1 & 0 \end{array}\right]$

... $c_1 = 1$, $c_2 = 3$

so IVP soln $y = e^{-3t} + 3te^{-3t}$