NIH Director

Differential equations. Francis Collins
life seemed to be reducible to second-order
graduate student studying quantum physics.
I became an atheist because, as a

Error: No section, no clue

This week we did no office hours
We're in the homestretch!

Linear DE
Homogeneous Second-Order

Lecture 20
Warmup 1) Find all solutions to $y' = ay$

$y = y(t)$

2) Find (unique) solution to $y' = ay$ such that $y(0) = 3$.

IVP = Initial Value Problem
Why 2 GM

Claim:

Suppose \( f \) is a soln

\[ y = ce^{at} \]

are all possible solns

\( y(0) = f(0) = a \)

Solution:

\[ y(t) = e^{at} \]
Want: $f = ce^{at}$ for some $c$.

Introduce $g = fe^{-at}$  
Expect $g$ consists.

$g' = f'e^{-at} + f'(-a)e^{-at}$

$= af'e^{-at} + (-a)fe^{-at}$

$= 0$

So $g = c$ some const.

Conclude $f = ce^{at}$
Set $c = 3$. Solve for $y$ if $y(0) = 3$.
A word about our fare, \( y = e^{at} \)

and any solution is a scale of it:

\[
\begin{align*}
z &= e^{at} + e^{(a+ib)t} \\
&= e^{at} \left(1 + e^{ibt}\right) \\
&= e^{at} \left(1 + \cos(bt) + i\sin(bt)\right)
\end{align*}
\]
Linear Algebra Interpretation \( V = \text{vector space of differential equations} \)

\[ T : V \rightarrow V \] linear transform \( f : \mathbb{R} \rightarrow \mathbb{R} \)

\[ T = \frac{\text{d}}{\text{d}t} \quad \text{so} \quad T(f) = \frac{\text{d}f}{\text{d}t}(t) = f' \]

Solve \( T(f) = a \) for \( a \)-vector equation!

Generl solution form \( \text{c-space} \quad E_a = \text{Span} \{ \text{e}^{at} \} \)

Initial Value Problem \( S : E_a \rightarrow \mathbb{R} \) solve linear system \( S(f) = f(0) \)
also IVP

h''(t) + q(t) + c h'(t) + y(t) = 0

h''(t) = h(t) = 0

h(0) = y(0) = \gamma

y(t) = y_h(t) + y_h(h(t)) + y_h''(t)

Now onto 2nd order ODE

✓ Non-Linear

✓ Homogeneous
Solve \( \text{S}(y) = \left[ y(0) \right] \) if \( \text{S}(y) = \left[ y(0) \right] \) is a vector in \( \mathbb{R}^2 \).

IVP given \( S : \text{null}(T) \rightarrow \mathbb{R}^2 \)

Find \( \text{null}(T) \)

Solve given \( T(y) = 0 \)

\[
T = \left( \frac{d^2}{dt^2} + \frac{d}{dt} + 4 \right)
\]

\[
T : \forall \rightarrow \forall
\]

Lin Alg Interpretation \( V = \text{vec span of } \)
Caution:

\[
\frac{x^2}{a^2 + b^2} \leq 1
\]
1) \( r_1 \neq r_2 \) gives real roots.

2) \( r_1 = r_2 \) repeated real root.

3) \( r_1 = r_2 = \alpha + i\beta \) complex conjugate roots.

Useful fact: auxiliary \( \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \)
Exer. 11 Find all solutions of

\[ y'' - 3y' - 4y = 0 \]

\[ y(0) = 0, \quad y'(0) = 1 \]

**EVP**

**Solution**

Aux eqn

\[ r^2 - 3r - 4 = 0 \]

Roots: \( r = 4, r = -1 \)

**Case (i)**

Distinct real roots

**Case (ii)**

One repeated real root

**Eigenvector**

\[ \psi_1 = (1, 1)' \]

\[ \psi_2 = (1, -1)' \]
Key idea: original ODE also factors!

\[(\frac{d}{dt} - 4) (\frac{d}{dt} - (-1)) y = 0\]

(order is unimportant)

Solutions of \( (\frac{d}{dt} - (-1)) y = 0 \) will also solve \( y'' - 3y' - 4y = 0 \)!

Similarly for \( (\frac{d}{dt} - 4) y = 0 \)
Fact

Any soln is of form

\[ y = c_1y_1 + c_2y_2 \]

where for

\[ z^2 + e^{2t} + e^{-t} = 0 \]
\[ z = e^{-t} - 1 \]

Find 2 dims of solns

\[ y_1 = \frac{a}{e^t} \]
\[ y_2 = \frac{b}{e^t} \]

Sups to spry 2
In other words

Solve of \( y'' - 3y' - 4y = 0 \)

form

Span \( \{ y_1 = e^t, y_2 = e^{-t} \} \)
$e^{4t} - 12e^{4t} - 4e^{4t} = 0$

Take $y = e^{4t}$

Plug in to $y - 3y' - 4y = 0$
Evaluate

\[ y(0) = c_1 y_1(0) + c_2 y_2(0) \]
\[ = 0 \]

Solve for \( c_1, c_2 \) so that \( y(0) = 0 \)

Now solve IVP

\[ y = c_1 y_1 + c_2 y_2 \]
\[ y_1' + 2 y_1 + y_2 = 0 \]
This is a linear system.

\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
= 0
\]
repeated real root (case (2))

\((r + 3) (r + 3) = 0\)

\(r^2 + 6r + 9 = 0\)

\(\text{Solve aux eqn}\)

\(y(0) = 1, \ y'(0) = 0\)

\(2) \text{ Solve IVP with}\)

\(y'' + 6y' + 9y = 0\)

\(0, 2 \text{ both all solutions}\)
Factor original ODE

\((\frac{d}{dt} + 3)(\frac{d}{dt} + 3)y = 0\)

We find a soln \(y_1 = e^{-3t}\)

Where is expected second soln?

Try \(y_2 = te^{-3t}\)

Check \(\frac{d}{dt} + 3)(\frac{d}{dt} + 3) te^{-3t}\)

\(= (\frac{d}{dt} + 3)(e^{-3t}) = 0\)
Lin Sys

\[
\begin{bmatrix}
-3 & 1 \\
-1 & 0 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
y_1(0) = 1 \\
y_2(0) = 2 \\
0 \\
\end{bmatrix}
\]

Solve IVP:

General soln

\[y = c_1 e^{-3t} + c_2 t e^{-3t} + 3 \]

\[y(0) = 1 \]
So IVP soln \( y = e^{-3t} + 3te^{-3t} \)

\( c_1 = 1, \quad c_2 = 3 \)