Welcome Back!

Lecture 17 Applications of Orthogonality

This week: Wed Office Hours, 12-2pm, 891 Evans

Thurs Midterm review, 2-3:30pm, 740 Evans

Fri Quiz through §7.1

Next week: Midterm 2, Tues during lecture meeting through §6.5

"Keep the flame of curiosity and wonderment alive, even when studying for boring exams." Michio Kaku

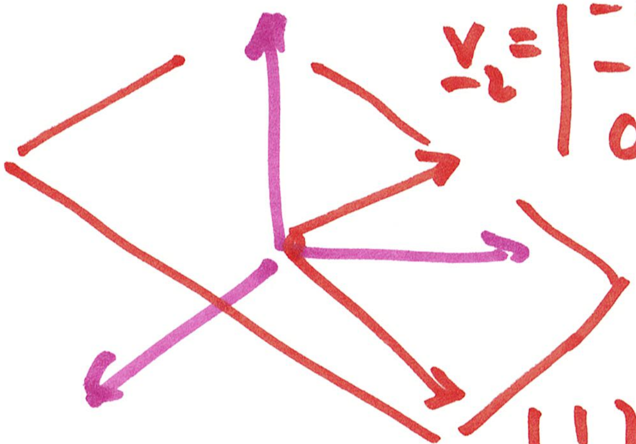
... And also, learn the math.

Warmup Find orthog basis for image of $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Soln

Image (A) = span of pivot cols.

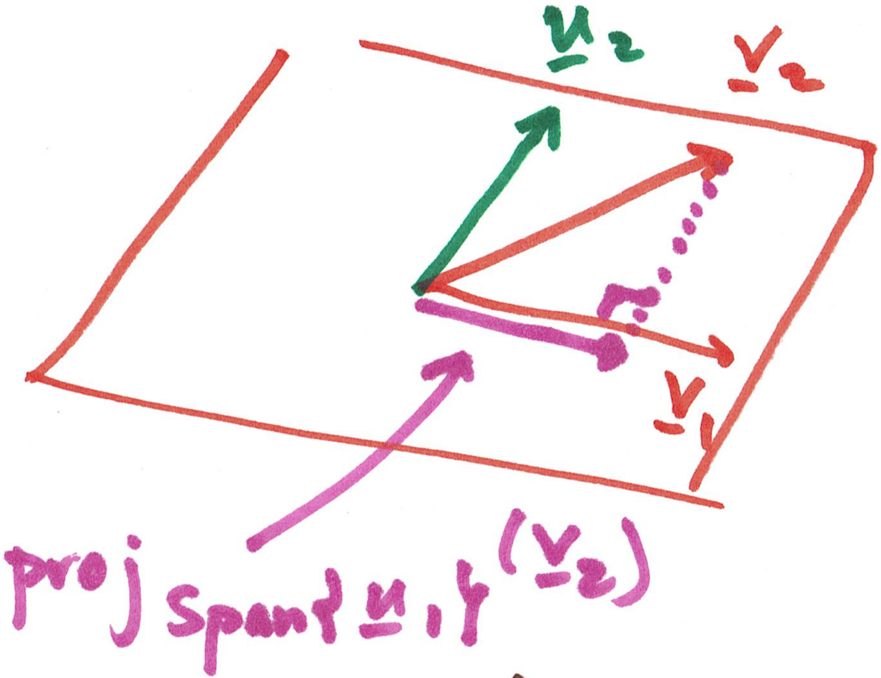


$$\underline{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

basis of image

Apply G-S



$$\underline{u}_1 = \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{u}_2 = \underline{v}_2 - \text{proj}_{\text{Span}\{\underline{u}_1\}}(\underline{v}_2)$$

$$= \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 = \dots \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

Organize G-S into $\underline{A = QR \text{ factorization}}$

General story given k lin indep vectors $\underline{v}_1, \dots, \underline{v}_k \in \mathbb{R}^n$

place in cols of $n \times k$ matrix

$$A = \underbrace{\begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{bmatrix}}_{k} \Big\} n$$

$$\xrightarrow[\sim]{\text{G-S}} \underline{u}_1, \dots, \underline{u}_k \in \mathbb{R}^n \xrightarrow[\sim]{\text{norm.}} \hat{\underline{u}}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|}, \dots, \underline{u}_k = \frac{\underline{u}_k}{\|\underline{u}_k\|}$$

orthogonal and nonzero

orthon. set

place in cols $Q = \begin{bmatrix} | & & | \\ \hat{\underline{u}}_1 & \cdots & \hat{\underline{u}}_k \\ | & & | \end{bmatrix}$

$\underset{n\times k}{A} = \underset{n\times k}{Q}\,\underset{k\times k}{R}$ factorization

records col ops that take Q back to A

Key pt: R will be upper Δ-ar since $\underline{v}_i \in \text{Span}\{\hat{\underline{u}}_1, \ldots, \hat{\underline{u}}_i\}$

Formula for R

Suppose
$A = QR$

Then $$\underset{\substack{\uparrow\\ k\times n}}{Q^T}\,\underset{\substack{\uparrow\\ n\times k}}{A} = \underset{\substack{\uparrow\\ k\times n}}{Q^T}\,\underset{\substack{\uparrow\\ n\times k}}{Q}\,\underset{\substack{\uparrow\\ k\times k}}{R}$$

So $$Q^TA = \underset{\substack{\uparrow\\ Q \text{ has orthon. cols}\\ \text{so } Q^TQ = I_k}}{I_k}\cdot R = R$$

[**Aside:** $\underset{k\times n}{\underset{\nearrow}{Q^T}}\,\underset{n\times k}{\underset{\uparrow}{Q}} = I_k$

But what about $\underset{n\times k}{\underset{\nearrow}{Q}}\,\underset{k\times n}{\underset{\nwarrow}{Q^T}}$?

Ex $Q = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad Q^TQ = [1\ 0 \cdots 0]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [1]$

$= I_1$

But $QQ^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}[1\ 0 \cdots 0] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{bmatrix}$

$\neq I_n$]

Back to initial exer

$$A = \begin{vmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{vmatrix}$$

$$Q = \begin{array}{c} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{3\sqrt{\frac{2}{3}}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{3\sqrt{\frac{2}{3}}} \\ \frac{1}{\sqrt{3}} & \frac{2}{3\sqrt{\frac{2}{3}}} \end{bmatrix} \\ \begin{array}{cc} \nearrow & \nearrow \\ \hat{\underline{u}}_1 & \hat{\underline{u}}_2 \end{array} \end{array}$$

$$R = Q^T A = \begin{bmatrix} \sqrt{3} & -\frac{2}{3}\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

What are entries of $\underline{R}$ in terms of G-S?

$$\begin{vmatrix} \sqrt{3} & -\frac{2}{3}\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}} \end{vmatrix}$$

$$\underline{v}_1 = \underbrace{\sqrt{3}}_{\|\underline{u}_1\|} \cdot \hat{\underline{u}}_1$$

$$\underline{v}_2 = -\frac{2}{3}\sqrt{3}\ \hat{\underline{u}}_1 + \underbrace{\sqrt{\frac{2}{3}}}_{\|\underline{u}_2\|}\ \hat{\underline{u}}_2$$

Least Squares Approx

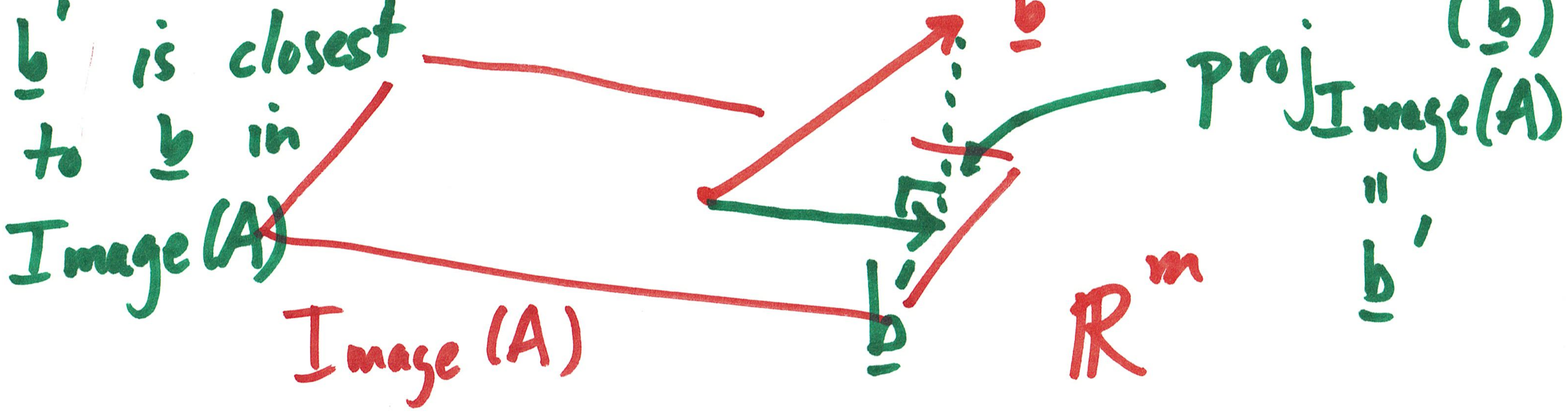
Suppose A $m \times n$ matrix

$\underline{b} \in \mathbb{R}^m$, we seek to solve

$A\underline{x} = \underline{b}$ for $\underline{x} \in \mathbb{R}^n$

Suppose $A\underline{x} = \underline{b}$ is inconsistent.

What is best we can do?



"Least squares" = minimizing

$$\| A\underline{x} - \underline{b} \|^2$$

Now solve $A\underline{x}' = \underline{b}'$

Could solve using usual methods...

But let's solve it
in one swell foop ☺

Observe 1) $(\underline{b} - \underline{b}') \perp \text{Image}(A)$

2) $\text{Image}(A) = \text{Col}(A) = \text{Row}(A^T)$

so $\text{Image}(A)^\perp = \text{Null}(A^T)$

1) + 2) $\implies \underline{b} - \underline{b}' \in \text{Null}(A^T)$

i.e. $A^T(\underline{b} - \underline{b}') = \underline{0}$

Suppose we solve $A\underline{x}' = \underline{b}'$

Calculate $A^T(\underline{b} - A\underline{x}')$

$$= A^T(\underline{b} - \underline{b}') = \underline{0}$$

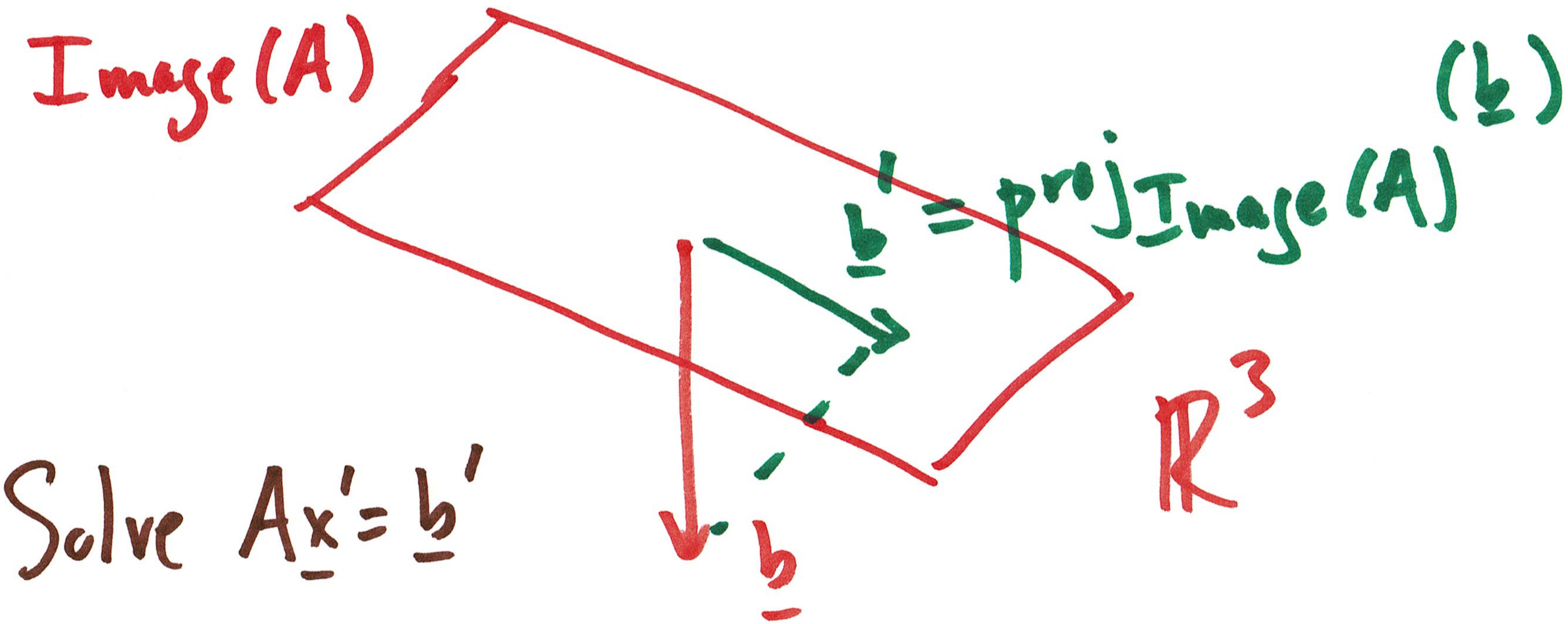
Rearrange $\underbrace{A^T A}_{C = A^T A}\underline{x}' = \underbrace{A^T \underline{b}}_{\underline{c} = A^T \underline{b}}$

Usual lin syst: $C\underline{x}' = \underline{c}$!

__Exer__ Find least squares soln to

$$A\underline{x} = \underline{b}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$



Solve $A\underline{x}' = \underline{b}'$

Soln Solve $C\underline{x}' = \underline{c}$

$C = A^T A \qquad \underline{c} = A^T \underline{b}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Solve $\left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right]$

$\ldots \quad \underline{x}' = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$

Note $\underline{b}' = A\underline{x}' = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

Another beautiful application
of orthogonality

Spectral Theorem Let A n×n matrix
(with real entries)

TFAE = the following are equivalent

1) A is symmetric : $A = A^T$

2) there is an orthog basis of e-vectors
for A

3) $D = P^{-1}AP$ for P orthog matrix
diagonalizable

But note A diagonalizable $\not\Rightarrow A = A^T$

for example $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

e-values $\lambda = 1, 3$ but

~~and~~ e-vectors will not be orthog.

Pf (2) $\Longrightarrow$ (3)

$\underline{v}_1, \ldots, \underline{v}_n$ orthog basis of e-vectors

then $\underline{u}_1 = \hat{\underline{v}}_1, \ldots, \underline{u}_n = \hat{\underline{v}}_n$ orthon. basis of e-vector

so $P = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}$ orthog. matrix

(3) $\Longrightarrow$ (2) $\underset{\text{diagonal}}{D} = P^{-1}A\underset{\text{orthog.}}{P}$ then cols of P form orthon basis of e-vectors.

(3) $\Rightarrow$ (1) $\underset{\substack{\uparrow \\ \text{diagonal}}}{D} = P^{-1} A \underset{\substack{\uparrow \\ \text{orthogonal}}}{P}$ so $A = PDP^{-1}$

$\underline{P \text{ orthogonal}} \Rightarrow P^{-1} = P^{T}$

$\underline{D \text{ diagonal} \Rightarrow D^{T} = D}$ so also $(P^{-1})^{T} = P$

Now calculate $A^{T} = (PDP^{-1})^{T}$

$= (P^{-1})^{T} D^{T} P^{T} = PDP^{-1} = A!$

so A symmetric!

Next time (1) $\Rightarrow$ (2) !

Stay tuned...