Lecture 15 — Geometry in $\mathbb{R}^n$!

This week: **Wed** Office Hrs, 12-2pm
891 Evans

Fri Quiz through §6.3

"Our offense is like the
Pythagorean Theorem:
there is no answer!"
Shaquille O'Neal

Last words on similarity, diagonalization

What is the "nicest" matrix similar to

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} ?$$

We seek: "good" basis β

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \arrow[d, "P_\beta^{-1}"] \\ \mathbb{R}^2 \arrow[u, "P_\beta"] \arrow[r, "C"] & \mathbb{R}^2 \end{tikzcd}$$

std coords

β-coords

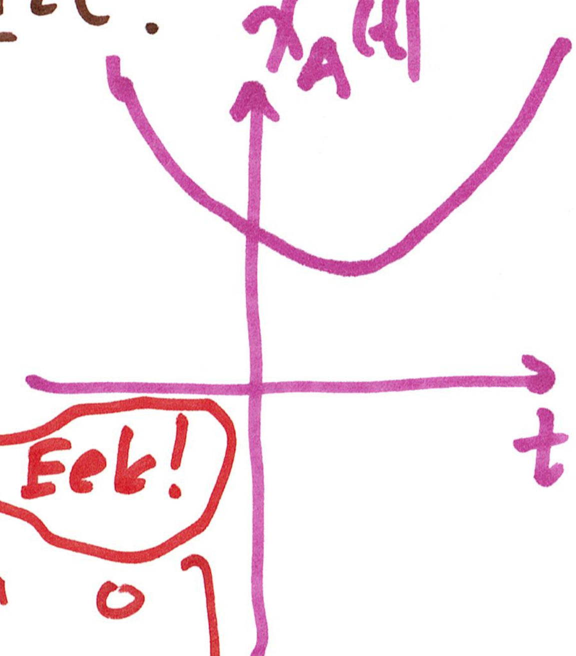
$$\boxed{C = P_\beta^{-1} A P_\beta}$$

How "nice" can we make C?

Best case: try to <u>diagonalize</u>!

<u>Step 1)</u> $\chi_A(t) = t^2 - 3t + 3$

no <u>real</u> roots!



Impossible to find $C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

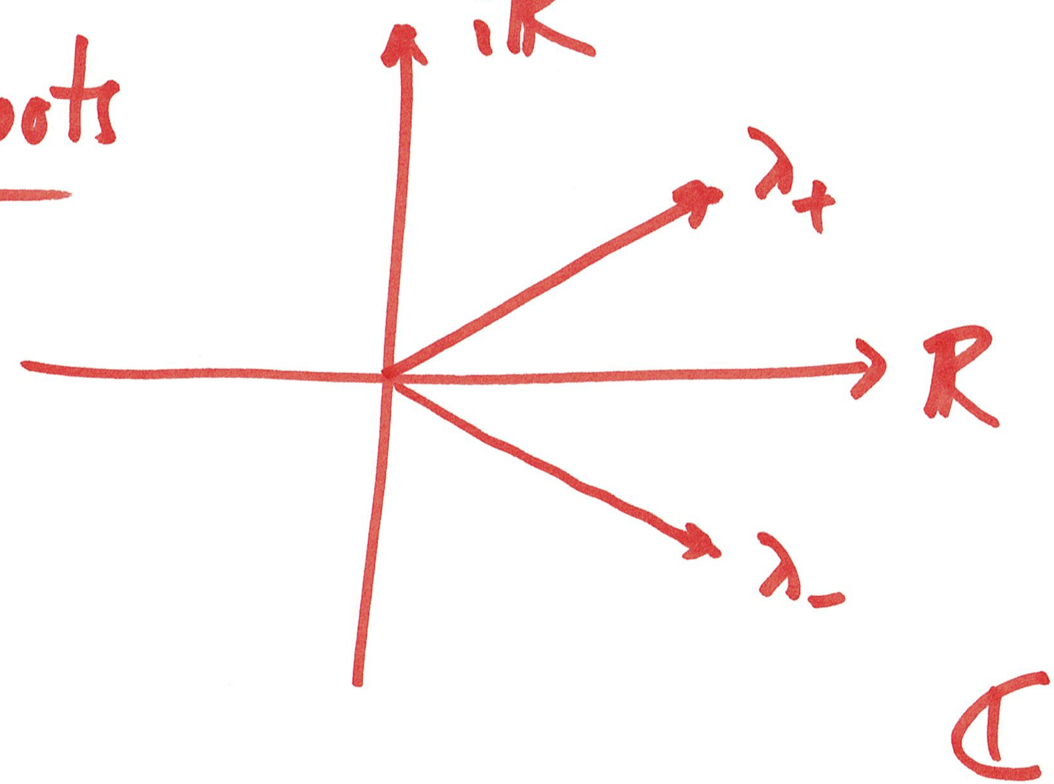
with λ_1, λ_2 <u>real</u>!

But... always can factor with complex roots

Quad formula: $\lambda_{\pm} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$

$$\chi_A(t) = (t - \lambda_+)(t - \lambda_-)$$

Picturing roots



Thm (**Fund Thm of Alg**)

Any poly $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

completely factors $p(t) = c(t-\lambda_1)\cdots(t-\lambda_n)$

if we allow complex roots $\lambda_1, \ldots, \lambda_n$!

Step 1) of finding roots of $\chi_A(t)$

can **never** fail if we

accept complex roots!

Back to exer:

Step 2) Take $C = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$

$P_\beta = \begin{bmatrix} | & | \\ \underline{v}_+ & \underline{v}_- \\ | & | \end{bmatrix}$ $\quad \beta = \{\underline{v}_+, \underline{v}_-\}$

e-vectors for λ_+, λ_-

We find: $\underline{v}_+ = \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$ $\quad \underline{v}_- = \begin{bmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$

by solving for $E_{\lambda_\pm} = \mathrm{Null}(A - \lambda_\pm I)$.

Suppose we're Neanderthals and don't accept complex numbers

What's the "nicest" possible C?

Introduce new basis $\gamma = \{\underline{w}_1, \underline{w}_2\}$

$$\underline{w}_1 = \mathrm{Re}(\underline{v}_+) = \frac{1}{2}\underline{v}_+ + \frac{1}{2}\underline{v}_- = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\underline{w}_2 = \mathrm{Im}(\underline{v}_+) = \frac{-i}{2}\underline{v}_+ + \frac{i}{2}\underline{v}_- = \begin{bmatrix} +\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \arrow[d, "P_\gamma^{-1}"] \\ \mathbb{R}^2 \arrow[u, "P_\gamma"] \arrow[r, "C"] & \mathbb{R}^2 \end{tikzcd}$$

$$C = P_\gamma^{-1} A P_\gamma$$

where $P_\gamma = \begin{bmatrix} | & | \\ \underline{w}_1 & \underline{w}_2 \\ | & | \end{bmatrix}$

We find: $C = \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}$

Is that so nice?!?

Yes! $C = \sqrt{3} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

scales by $\sqrt{3}$

rotates by $-\frac{\pi}{6}$

Never diag. with real e-values

<u>In general</u> we find

$$C = r \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$a = r\cos\theta \qquad b = r\sin\theta$$

We find ~~and~~ a, b from

$$\lambda_+ = a + ib \;!$$

<u>Now Geometry in $\mathbb{R}^n$</u>

Up to now we've never discussed <u>length</u> of vectors, <u>angles</u> between vectors (though we've discussed area, volume, ...)

Def Dot product / standard inner prod / Euclidean inner prod of vectors ~~for~~ $\underline{u}, \underline{v} \in \mathbb{R}^n$ is

$$\underline{u} \cdot \underline{v} = u_1 v_1 + \cdots + u_n v_n$$

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Satisfies many properties

Example: $\underline{v} \cdot \underline{v} = 0$

$$\Updownarrow$$

$$\underline{v} = \underline{0}.$$

Matrix mult interpretation

$$\underline{u} \cdot \underline{v} = \underline{u}^T \cdot \underline{v}$$

$$= \underbrace{\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{n \times 1}$$

$$= \underbrace{\begin{bmatrix} u_1 v_1 + \cdots + u_n v_n \end{bmatrix}}_{1 \times 1}$$

<u>Def</u> <u>Length</u> $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$

$$= \sqrt{u_1^2 + \cdots + u_n^2}$$

<u>Def</u> <u>Unit vector</u> $\underline{u}$ has length 1

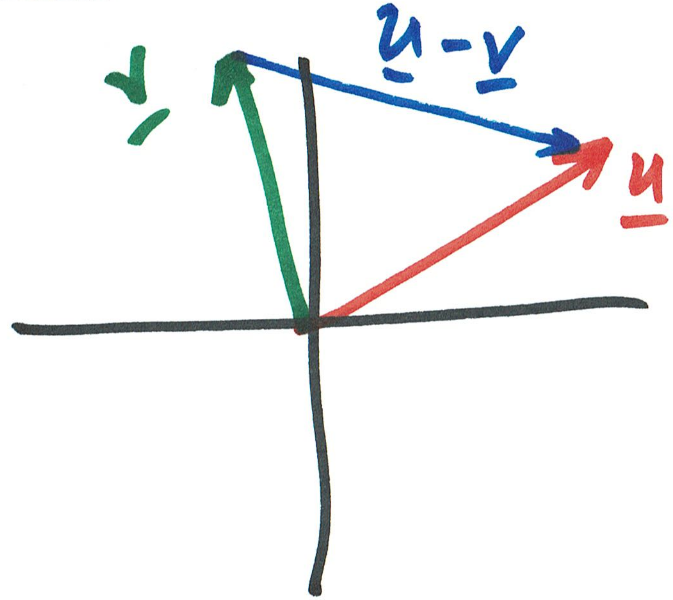
$$\|\underline{u}\| = \|\underline{u}\|^2 = 1$$

<u>Def</u> <u>Normalization</u> of $\underline{u} \neq \underline{0}$ is unit vector in ray through $\underline{u}$.

$$\hat{\underline{u}} = \frac{1}{\|\underline{u}\|} \cdot \underline{u}$$

Def Distance $d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$

$= \|\underline{v} - \underline{u}\| = d(\underline{v}, \underline{u})$



$\mathbb{R}^n$

But what does $\underline{u} \cdot \underline{v}$ mean geometrically?

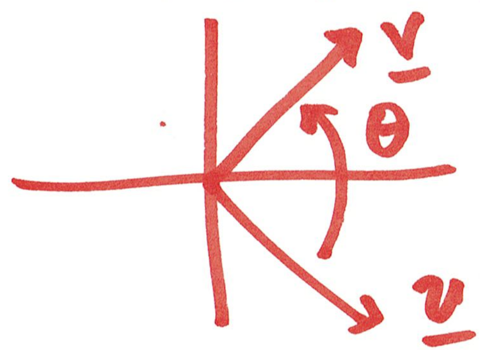
Let's calculate:

$$\|\underline{u}-\underline{v}\|^2 = (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})$$

$$= \underline{u}\cdot\underline{u} + \underline{v}\cdot\underline{v} - 2\,\underline{u}\cdot\underline{v}$$

Law of cosines imples

$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta$$

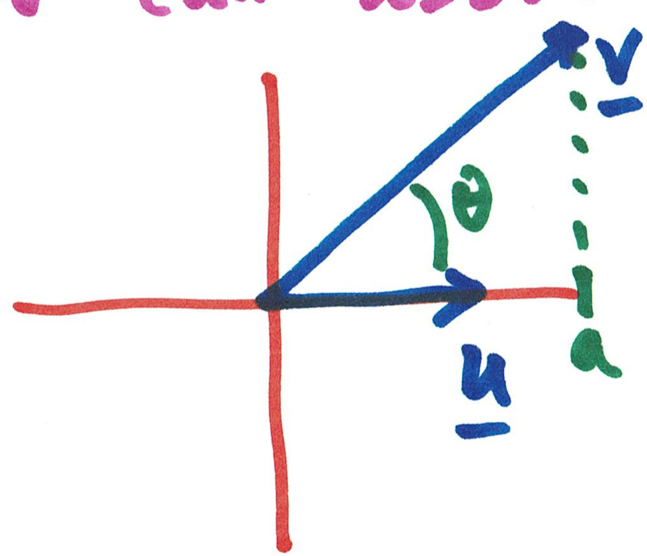


<u>Why</u> <u>true</u>? (at least for vectors in $\mathbb{R}^2$)

<u>Observe</u>
Formula is true $\iff$ true after scaling and rotating

So can assume $\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \underline{u} \cdot \underline{v} = a$

$\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad = \|\underline{v}\| \cos\theta$



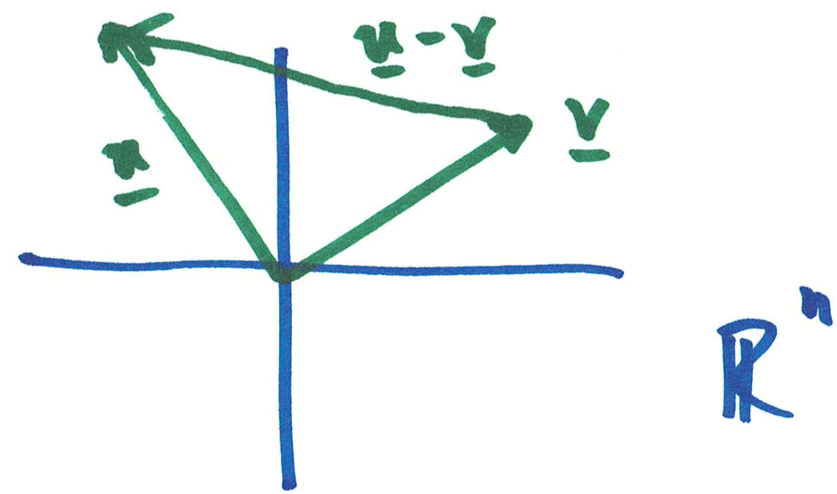
Def $\underline{u} \perp \underline{v}$ <u>orthogonal</u> means

$$\underline{u} \cdot \underline{v} = 0$$

(either $\underline{u}$ or $\underline{v}$ is $\underline{0}$, or angle between them is $\pm\frac{\pi}{2}$)

Pythagorean Thm $\underline{u} \perp \underline{v} \iff$

$$\|\underline{u}-\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$$



Pf $\|\underline{u}-\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - \underbrace{2\underline{u}\cdot\underline{v}}_{=0}$

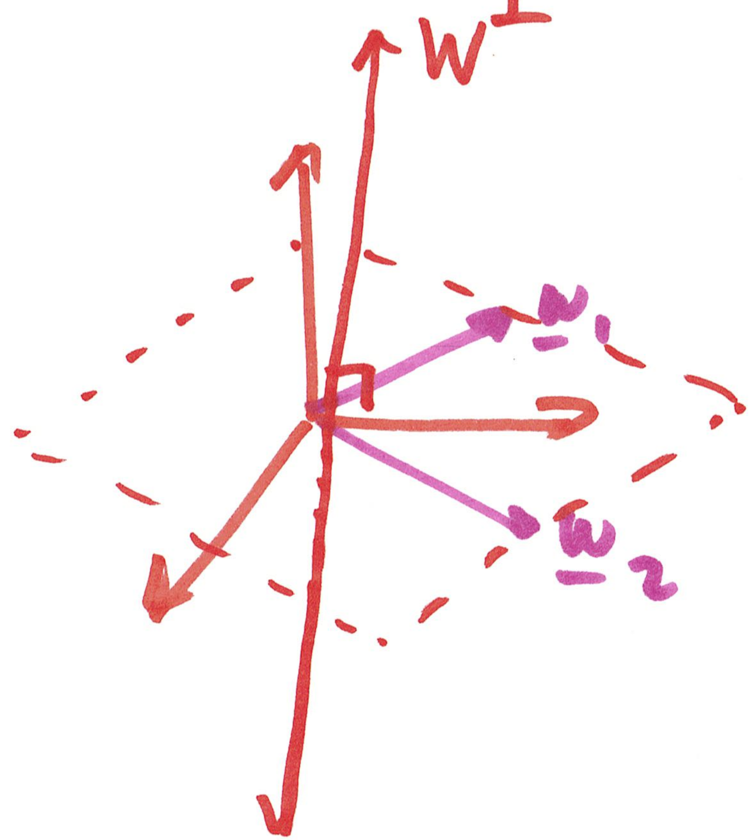
$\underline{u} \perp \underline{v} \iff$

∎

Def. $W \subset \mathbb{R}^n$ subset (not nec subspace)

orthogonal complement $W^\perp = \{ \underline{v} \in \mathbb{R}^n \mid \underline{v} \perp \underline{w}$ for all $\underline{w} \in W \}$

Cartoon



$W = \{ \underline{w}_1, \underline{w}_2 \}$

Exer 1) $W^\perp$ is a subspace of $\mathbb{R}^n$

(no matter what W is)

2) If W is a subspace, then

$$(W^\perp)^\perp = W$$

3) If $W = \text{Span}\{\underline{w}_1, \dots, \underline{w}_k\}$

then $W^\perp = \{\underline{w}_1, \dots, \underline{w}_k\}^\perp$

<u>Exer</u> A $m \times n$ matrix

Show $\mathrm{Row}(A)^{\perp} = \mathrm{Null}(A)$.

<u>Def</u> 1) $\underline{v}_1, \dots, \underline{v}_k$ <u>orthogonal</u> if $\underline{v}_i \perp \underline{v}_j$ all $i \neq j$.

2) $\underline{v}_1, \dots, \underline{v}_k$ <u>orthonormal</u> if orthogonal and $\|\underline{v}_i\| = 1$ all i.

3) $\underline{v}_1, \dots, \underline{v}_n$ <u>orthog.</u> <u>basis</u> if orthog. and a basis

4) $\underline{v}_1, \dots, \underline{v}_n$ <u>orthon. basis</u> if orthon. and a basis

<u>Thm</u> $\underline{v}_1, \dots, \underline{v}_k$ orthog and $\underline{v}_i \neq \underline{0}$ all i

then $\underline{v}_1, \dots, \underline{v}_k$ lin indep.

<u>Pf.</u> Suppose $a_1 \underline{v}_1 + \dots + a_k \underline{v}_k = \underline{0}$

Take dot prod with $\underline{v}_i$ to get.

$$a_i \, \underline{v}_i \cdot \underline{v}_i = 0$$

$$\shortparallel$$

$$a_i \|\underline{v}_i\|^2 \quad \text{so} \quad a_i = 0 \quad \boxtimes$$

$\neq 0$

Exer Is following orthog basis? orthon. basis?

$$\underline{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

Soln Check that $\underline{v}_i \cdot \underline{v}_j = 0$ for $i \neq j$

So they form a set of orthog, nonzero vectors.

By Thm, they are lin indep

so a basis.

in fact orthog basis.

But $\|\underline{v}_1\| = \sqrt{5}$ so <u>not</u> orthon. basis.

Why do we **love** orthog bases?

Exer Find coords of $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

w.r.t. basis of prior exer.

This means find a_1, a_2, a_3 so that

$$\underline{v} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3$$

Traditional method: solve lin syst.

$A \underline{x} = \underline{v}$ where $A = \begin{bmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{bmatrix}$ $\underline{x} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Suppose $\underline{v} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3$

Take dot product with $\underline{v}_i$

$$\underline{v}_i \cdot \underline{v} = a_i \, \underline{v}_i \cdot \underline{v}_i \quad !$$

Conclude $a_i = \dfrac{\underline{v}_i \cdot \underline{v}}{\underline{v}_i \cdot \underline{v}_i}$ nice formula!

$\underline{\text{Ex}}$ $a_1 = \dfrac{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} = \dfrac{2}{5}$