Welcome to Lecture 14!

Similarity of Matrices

Fri Quiz through §5.4

"Love is the power to see similarity in the dissimilar."

– Theodor Adorno

Def Two $n \times n$ matrices A, B are **similar** $A \sim B$ if there is invertible matrix P such that

$$B = P A P^{-1}$$

"A and B represent the same transf $T : \mathbb{R}^n \to \mathbb{R}^n$ but w.r.t. different bases"

<u>Caution</u> This is different from row equiv

$$B = EA \qquad \underline{\text{row equiv}}$$

for E an invertible matrix

<u>Observe</u> 3 standard properties of similarity

1) $A \sim A$ $\quad (A = IAI^{-1})$

2) $A \sim B \iff B \sim A$

$(B = PAP^{-1} \iff A = QBQ^{-1}$ for $Q = P^{-1})$

3) $A \sim B$, $B \sim C \implies A \sim C$ $\quad$ (Exer...)

Thm $A \sim B$ similar

then 1) $\chi_A(t) = \chi_B(t)$

$\Rightarrow$ same e-values, same multiplicities

2) $\dim E_\lambda$ is same for A, B

$\not\Rightarrow$ same e-vectors

$\Rightarrow$ A has basis of e-vectors $\Leftrightarrow$ B has basis of e-vectors

Pf of 1) $B = PAP^{-1}$ since $A \sim B$

Consider $P(A - t \cdot I)P^{-1}$

Take det to get

$$\det(PAP^{-1} - tI) = \det(P(A - tI)P^{-1})$$

$$\chi_B(t) = \det(PAP^{-1} - tI)$$

$$\det(P(A - tI)P^{-1}) = \det(P)\det(A - tI)\det(P^{-1}) = \chi_A(t)$$

__Exer__ Prove 2)

__Caution__ Converse is __not__ true

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \not\sim B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\chi_A(t) = t^4, \lambda = 0$ $\qquad$ $\chi_A(t) = t^4, \lambda = 0$

$\dim E_0 = 2$ $\qquad$ $\dim E_0 = 2$

Seeing $A \nsim B$ is not so easy...

Take Math 110 ! 

Learn Jordan canonical form

Our tools are sufficient to say whether diagonalizable matrices are similar or not.

<u>Thm</u> If A, B are diagonalizable

and 1) $\chi_A(t) = \chi_B(t)$

2) $\dim E_\lambda$ is same for A, B

Then $A \sim B$.

<u>In fact</u> If A, B are diagonalizable

then 1) $\Rightarrow$ 2).

__Pf.__ A, B diagonalizable

$\Rightarrow$ there are invertible P, Q

so that $PAP^{-1} = C$ ← diagonal

$QBQ^{-1} = D$ ← diagonal

$$C = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad D = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$$

__Exer__ Show there is invertible R so that $C = RDR^{-1}$!

Use 1) & 2) hypotheses.

We conclude:

$$A \sim C \sim D \sim B$$

so $\quad A \sim B$!

Exer Is $A \sim B$?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1/2 & 1/4 \\ 3 & 3/2 \end{bmatrix}$$

Soln $\chi_A(t) = (-t)(2-t)$ ☺

$\chi_B(t) = (-t)(2-t)$

$\Longrightarrow$ same e-values

$\lambda = 0, 2$

both of mult $= 1$

Observe $\dim E_0 = \dim E_2 = 1$ for both A, B

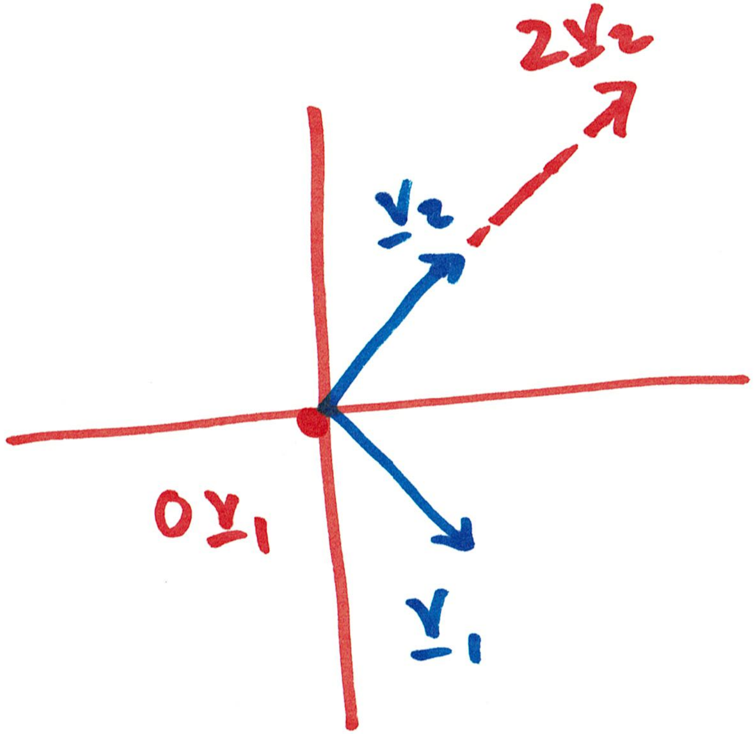
since mult of $0, 2$ are 1.

So both A, B have bases of e-vectors

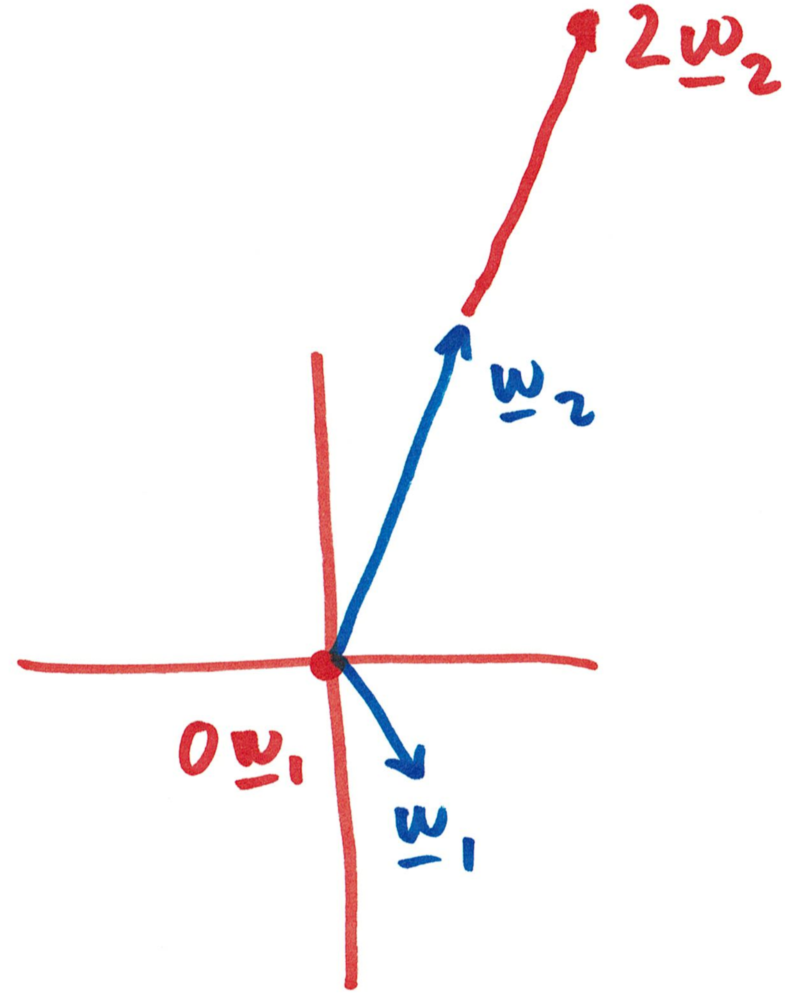
$$\alpha = \left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\beta = \left\{ \underline{w}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \underline{w}_2 = \begin{bmatrix} 2 \\ 12 \end{bmatrix} \right\}$$

Picture



$2\underline{v}_2$
$\underline{v}_2$
$0\underline{v}_1$
$\underline{v}_1$



$2\underline{w}_2$
$\underline{w}_2$
$0\underline{w}_1$
$\underline{w}_1$

We know already $A \sim B$ since
we know A, B diagonalizable
with same e-values
and dims of e-spaces.

But still fun to find P so that

$$B = PAP^{-1}$$

"Think of P as change of coords
matrix"

$$\begin{tikzcd}
V = \mathbb{R}^2 \arrow[r, "T = A"] \arrow[d, "T_\alpha"'] \arrow[dr, dashed, bend left] & V = \mathbb{R}^2 \arrow[d, "T_\alpha"] \\
\mathbb{R}^2 \arrow[r, "{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}"'] & \mathbb{R}^2
\end{tikzcd}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = T_\alpha A T_\alpha^{-1}$$

"α-matrix of A"

$$\begin{tikzcd}
V = \mathbb{R}^2 \arrow[r, "S = B"] \arrow[d, "T_\beta"'] \arrow[dr, dashed, bend left] & V = \mathbb{R}^2 \arrow[d, "T_\beta"] \\
\mathbb{R}^2 \arrow[r, "{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}"'] & \mathbb{R}^2
\end{tikzcd}$$

"β-matrix of B"

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = T_\beta B T_\beta^{-1}$$

Altogether:

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "A"] \arrow[d, "P_\alpha^{-1} = T_\alpha"'] & \mathbb{R}^2 \arrow[d, "T_\alpha = P_\alpha^{-1}"] \arrow[dd, bend left=60, "P"] \\
\mathbb{R}^2 \arrow[r, "\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}"] & \mathbb{R}^2 \\
\mathbb{R}^2 \arrow[r, "B"] \arrow[u, "P_\beta^{-1} = T_\beta"] \arrow[uu, bend left=60, "P^{-1}"] & \mathbb{R}^2 \arrow[u, "T_\beta = P_\beta^{-1}"']
\end{tikzcd}
$$

Conclude $B = T_\beta^{-1} T_\alpha A T_\alpha^{-1} T_\beta$

$= P A P^{-1}$

What is $P = T_\beta^{-1} T_\alpha$ in terms of coord change matrices?

So $$P = P_\beta \, P_\alpha^{-1}$$

where ${}_{\varepsilon\leftarrow}P_\alpha$ changes α-coords to std coords

${}_{\varepsilon\leftarrow}P_\beta$ changes β-coords to std coords.

Now let's calculate

$$P = P_\beta P_\alpha^{-1}$$

for
$$P_\alpha = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$P_\beta = \begin{bmatrix} | & | \\ \underline{w}_1 & \underline{w}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 12 \end{bmatrix}$$

So $P_\alpha^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

<u>Conclude</u> $P = \begin{bmatrix} 1 & 2 \\ -2 & 12 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 10 & 14 \end{bmatrix}$$