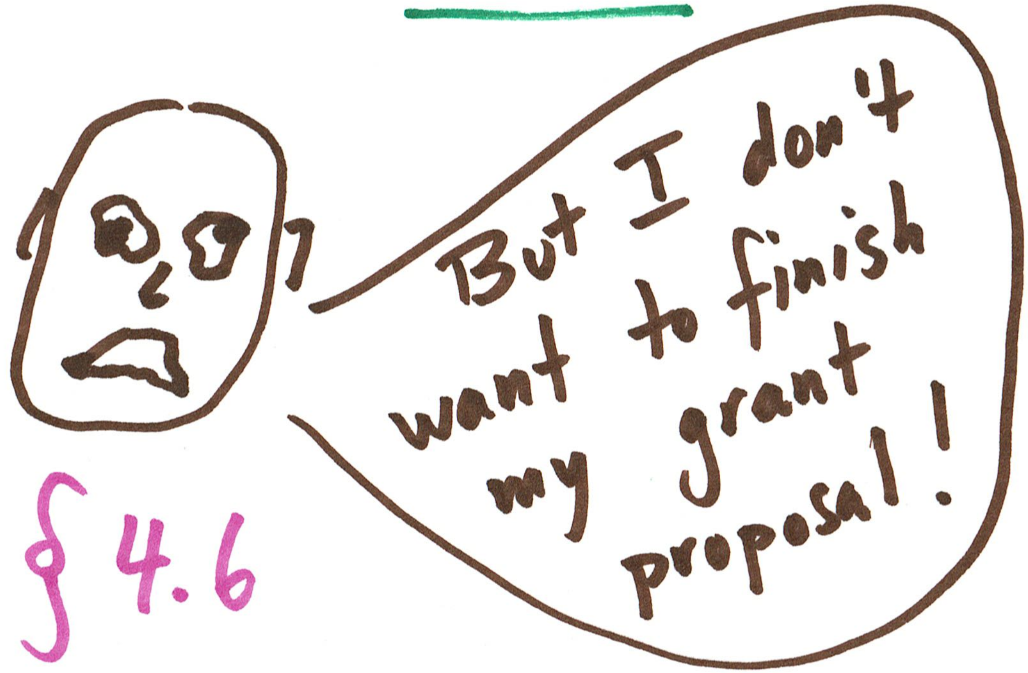
Welcome to Lecture 11!

Changes of Coordinates

Wed No Office Hours this week



Fri Quiz through §4.6

"Everything should be made as simple as possible, but not simpler."

Albert Einstein

Abstract Challenge

Suppose $T: V \to W$ is lin transf, and $\dim V = n < \infty$
$\dim W = m < \infty$

Suppose $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ is basis of V
$\gamma = \{\underline{w}_1, \dots, \underline{w}_m\}$ is basis of W

$\underline{w} = c_1\underline{w}_1 + \dots + c_m\underline{w}_m$

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "T_\gamma"] \\ \mathbb{R}^n \arrow[u, "T_\beta^{-1}"] \arrow[r] & \mathbb{R}^m \end{tikzcd}$$

Find $m \times n$ matrix A

$a_1\underline{v}_1 + \dots + a_n\underline{v}_n = T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)$

$T_\gamma(\underline{w}) = [\underline{w}]_\gamma = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$

Soln

$$A = \begin{bmatrix} | & & | \\ [T(\underline{v}_1)]_\gamma & \cdots & [T(\underline{v}_n)]_\gamma \\ | & & | \end{bmatrix}$$

$$\begin{tikzcd} \underline{v}_i \arrow[r, mapsto] & T(\underline{v}_i) \arrow[d, mapsto] \\ \underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \arrow[u, mapsto] & [T(\underline{v}_i)]_\gamma \end{tikzcd}$$

<u>Special case</u> $V = W$, $T = I$ = identity map = "do nothing"

Then $A = \begin{bmatrix} | & & | \\ [\underline{v}_1]_\gamma & \cdots & [\underline{v}_n]_\gamma \\ | & & | \end{bmatrix}$

$$\begin{tikzcd} & V \arrow[dr, "T_\gamma"] & \\ \mathbb{R}^n \arrow[ur, "T_\beta^{-1}"] \arrow[rr, "A"] & & \mathbb{R}^n \end{tikzcd}$$

__Def__ __Change of coord matrix__

from basis $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ of V

to basic $\gamma = \{\underline{w}_1, \dots, \underline{w}_n\}$

$$\underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} | & & | \\ [\underline{v}_1]_\gamma & \cdots & [\underline{v}_n]_\gamma \\ | & & | \end{bmatrix}$$

Similarly

$$\underset{\beta \leftarrow \gamma}{P} = \begin{bmatrix} | & & | \\ [\underline{w}_1]_\beta & \cdots & [\underline{w}_n]_\beta \\ | & & | \end{bmatrix}$$

Change of coords matrices satisfy

$$\underset{\gamma \leftarrow \beta}{P} [\underline{v}]_\beta = [\underline{v}]_\gamma \,, \quad \underline{v} \in V$$

$$\underset{\beta \leftarrow \gamma}{P} [\underline{v}]_\gamma = [\underline{v}]_\beta \,, \quad \underline{v} \in V$$

<u>Note</u> $\left(\underset{\gamma \leftarrow \beta}{P}\right)^{-1} = \underset{\beta \leftarrow \gamma}{P}$

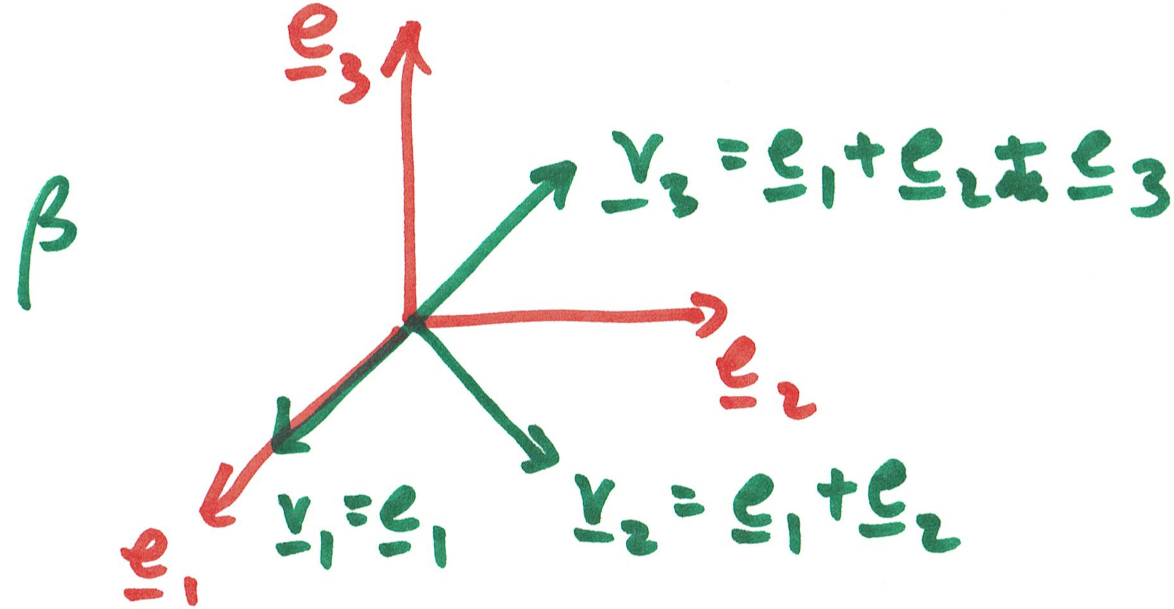
$$\left(\underset{\beta \leftarrow \gamma}{P}\right)^{-1} = \underset{\gamma \leftarrow \beta}{P}$$

__Exer__ Take $V = \mathbb{R}^3$

Consider bases

$$\beta = \left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\gamma = \left\{ \underline{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \underline{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$



Convert following β-coords to γ-coords

1) $[\underline{v}]_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \underline{v} = 0 \cdot \underline{v}_1 + 0\,\underline{v}_2 + 1 \cdot \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

these are coords of $\underline{v}$ wrt std basis!

$[\underline{v}]_\gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ since $\underline{v} = 0 \cdot \underline{w}_1 + 0 \cdot \underline{w}_2 + 1\,\underline{w}_3$

2) $[\underline{v}]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\qquad \underline{v} = 0 \cdot \underline{v}_1 + 1 \cdot \underline{v}_2 + 0 \cdot \underline{v}_3$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

again these are coords of $\underline{v}$ wrt std basis

$[\underline{v}]_\gamma = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that

$$\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = a \cdot \underline{w}_1 + b\,\underline{w}_2 + c\,\underline{w}_3$$

$$= a \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

this is a lin syst

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \quad [\underline{v}]_\gamma = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

3) $[\underline{v}]_\beta = \begin{bmatrix} 2017 \\ e \\ \pi \end{bmatrix}$

$[\underline{v}]_\gamma = \; ? \; = \underset{\gamma \leftarrow \beta}{P} [\underline{v}]_\beta$

$$\underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} | & | & | \\ [\underline{v}_1]_\gamma & [\underline{v}_2]_\gamma & [\underline{v}_3]_\gamma \\ | & | & | \end{bmatrix}$$

We need to solve <u>3 lin</u> sys<u>ts</u>!

$$\underline{v}_1 = a_1 \cdot \underline{w}_1 + b_1 \underline{w}_2 + c_1 \underline{w}_3$$

$$\underline{v}_2 = a_2 \underline{w}_1 + b_2 \underline{w}_2 + c_2 \underline{w}_3$$

$$\underline{v}_3 = a_3 \underline{w}_1 + b_3 \underline{w}_2 + c_3 \underline{w}_3$$

$$\left[\begin{array}{ccc|ccc} | & | & | & | & | & | \\ \underline{w}_1 & \underline{w}_2 & \underline{w}_3 & \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | & | & | & | \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c:c} I_3 & \underset{\gamma \leftarrow \beta}{P} \end{array} \right]$$

Conclude

$$[\underline{v}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\underline{v}]_\beta = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2017 \\ e \\ \pi \end{bmatrix}$$

$$= \begin{bmatrix} -e \\ -2017 \\ 2017+e+\pi \end{bmatrix} !$$

Suppose V has 3 bases β, γ, δ

$$\begin{tikzcd}
& V \arrow[dl, "T_\beta"'] \arrow[d, "T_\gamma"] \arrow[dr, "T_\delta"] & \\
\mathbb{R}^n \arrow[r, bend left, "\underset{\gamma \leftarrow \beta}{P}"'] \arrow[rr, bend left=80, "\underset{\delta \leftarrow \beta}{P}"] & \mathbb{R}^n \arrow[r, bend left, "\underset{\delta \leftarrow \gamma}{P}"'] & \mathbb{R}^n
\end{tikzcd}$$

$$\underset{\delta \leftarrow \beta}{P} = \underset{\delta \leftarrow \gamma}{P} \; \underset{\gamma \leftarrow \beta}{P} \quad !$$

__Special case__ $V = \mathbb{R}^n$, γ = std basis

$\beta_{std} = \{\underline{e}_1, \dots, \underline{e}_n\}$

$$\underset{\gamma \leftarrow \beta}{P} = \underset{\gamma \leftarrow \beta_{std}}{P} \; \underset{\beta_{std} \leftarrow \beta}{P}$$

__Notation__ $$P_\beta = \underset{\beta_{std} \leftarrow \beta}{P} = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$$

where $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$

Conclusion $\underset{\delta \leftarrow \beta}{P} = P_\delta^{-1} P_\beta$

$$\left[\begin{array}{c|c} P_\delta & P_\beta \end{array}\right] \rightsquigarrow \left[\begin{array}{c|c} I_n & P_\delta^{-1} P_\beta \end{array}\right]$$

__Exer__ Suppose $\beta = \left\{ \underline{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \underline{v}_3 \right\}$

basis for $\mathbb{R}^3$

Find $\underline{v}_3$ so that $\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

__Soln__ $1 \cdot \underline{v}_1 + 2 \cdot \underline{v}_2 + 3 \cdot \underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} -1 + 2 + 3x \\ 2 + 2 + 3y \\ -1 + 4 + 3z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 3x \\ 3y \\ 3z \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -3 \end{bmatrix} \rightsquigarrow \underline{v}_3 = \begin{bmatrix} 0 \\ -4/3 \\ -1 \end{bmatrix}$$