1. (5 points) What are the dimensions of the null space and column space of

\[ A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & 1 & 8 \\ 3 & 3 & 9 \end{bmatrix} \]

**Solution:** Augment \( A \) with the 0 vector and row reduce to find the dimension of the vector space of solutions to the equation \( Ax = 0 \). Row reducing, we obtain

\[ \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

So, the dimension of the vector space of solutions to the equation \( Ax = 0 \) is 1. Hence, the dimension of the null space of \( A \) is 1. By the rank theorem, the dimension of the column space of \( A \) is equal to \( 2 = 3 - 1 \).
2. (5 points) If an $m \times n$ matrix $A$ has rank $k$, find the dimension of the null space of $A^T$.

Solution:
The column space of $A^T$ is the row space of $A$, which has same dimension as the column space of $A$, which is $k$. By the rank theorem applied to $A^T$,

$$m = \dim \text{Nul } A^T + k$$

So,

$$\dim \text{Nul } A^T = m - k$$