You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ______________________________

1. (5 points) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -1 & 1 & 2 \\ 2 & 5 & 7 & 12 \end{bmatrix}.$$

Solution: The space $\operatorname{Nul} A$ is the collection of all $x \in \mathbb{R}^4$ such that $Ax = 0$. The standard method is to reduce A into RREF and find the general solution in vector form.

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -1 & 1 & 2 \\ 2 & 5 & 7 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & -3 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 & 11 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By the above calculation, we know the general solution has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6t - 11s \\ t + 2s \\ t \\ s \end{bmatrix}$$

where $s, t \in \mathbb{R}$. In vector form, the general solution is $tb_1 + sb_2$ where $b_1 = \begin{bmatrix} -6 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -11 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ and so a basis for $\operatorname{Nul} A$ is $\mathcal{B} = \{b_1, b_2\}$.

2. (5 points) Use an inverse matrix to find $[x]_{\mathcal{B}}$ for the vector $x \in \mathbb{R}^2$ and basis $\mathcal{B}$ of $\mathbb{R}^2$ given below.

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \end{bmatrix} \right\} \qquad x = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

Solution: We recall that by definition

$$x = P_{\mathcal{B}}[x]_{\mathcal{B}}.$$

where $P_{\mathcal{B}} = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$ is the change of basis matrix from $\mathcal{B}$ to the standard basis. By multiplying by $P_{\mathcal{B}}^{-1}$, the equation above is equivalent to

$$[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}x.$$

We are now left to compute $P_{\mathcal{B}}^{-1}$ and multiply. We have

$$\left[\begin{array}{cc|cc} 3 & -3 & 1 & 0 \\ 2 & -4 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{3} & 0 \\ 1 & -2 & 0 & \frac{1}{2} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{3} & 0 \\ 0 & -1 & -\frac{1}{3} & \frac{1}{2} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{2} \end{array}\right].$$

so that

$$P_{\mathcal{B}}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}.$$

Finally, we get

$$[x]_{\mathcal{B}} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$