Name: ____________________________

1. (5 points) Determine if the set $D = \{ f \colon \mathbb{R} \to \mathbb{R} \mid f \text{ is once differentiable} \}$ of once differentiable functions is a subspace of the vector space $V = \{ f \colon \mathbb{R} \to \mathbb{R} \}$ of all functions on $\mathbb{R}$.

Solution: Since differentiable functions are functions, $D \subseteq V$. Then we need to check three things to determine if $D$ is a subspace of $V$: first, we must check that the zero vector is in $D$; second, we must check that $D$ is closed under vector addition; and third, we must check that $D$ is closed under multiplication by scalars.

The zero vector in $V$ is the zero function $f(x) = 0$, which is certainly differentiable. Thus, $0 \in D$.

Next, suppose that $f, g \in D$; i.e., that $f$ and $g$ are two differentiable functions. Our knowledge of calculus I tells us that if $f$ and $g$ are differentiable, so is $f + g$. Hence, $f + g \in D$.

Finally, let $f \in D$ and let $c \in \mathbb{R}$. Again, calculus I tells us that if $f$ is differentiable and $c$ is a constant, so is $cf$.

Thus, all three properties are satisfied, and $D$ is a subspace of $V$.

2. (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 10 \\ -3 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute the volume of $T(B)$, where $B$ is the box $-1 \leq x_1 \leq 2$, $0 \leq x_2 \leq 1$, $1 \leq x_3 \leq 3$.

Solution: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 10 \\ -3 & -3 & 1 \end{bmatrix}$, the volume of $T(B)$ is equal to $|\det(A)| \cdot \Vol(B)$.

$B$ is a parallelepiped with sides parallel to the axis, so its volume is

$$\Vol(B) = (2 - (-1)) \cdot (1 - 0) \cdot (3 - 1) = 6.$$ 

Next we compute the determinant of $A$. If we subtract 2 times row 1 from row 2 and add 3 times row 1 to row 3, we do not change the determinant of $A$, so we may instead compute the determinant of $A' = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 0 \\ 0 & 6 & 16 \end{bmatrix}$.

Using the cofactor expansion in the first column, we see that

$$\det(A') = 1 \cdot \det \begin{bmatrix} 2 & 0 \\ 6 & 16 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 3 & 5 \\ 6 & 16 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix} = 1 \cdot 2 \cdot 16 - 0 + 0 = 32.$$ 

Thus, the volume of $T(B)$ is

$$\Vol(T(B)) = |\det(A)| \cdot \Vol(B) = |\det(A')| \cdot 6 = 32 \cdot 6 = 192.$$