You have 20 minutes to complete this quiz. To receive full credit, you must justify your answers.

Name : ______________________________

1. (5 points) Find the general solution of the linear system corresponding to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 4 & 0 & 2 \\ 1 & 1 & -2 & 2 \end{array}\right]$$

Solution: We apply row operations to get the matrix to row echelon form

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 4 & 0 & 2 \\ 1 & 1 & -2 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & -4 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We may go further and get to reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_2/2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the system

$$\begin{aligned} x_1 - 4x_3 &= 3 \\ x_2 + 2x_3 &= -1 \end{aligned}$$

We may then write the general solution as

$$\begin{aligned} &x_1 = 4x_3 + 3 \\ &x_2 = -2x_3 - 1 \\ &x_3 \text{ free} \end{aligned}$$

2. (5 points) Find all values of c for which the vector $\begin{bmatrix} 3 \\ 3 \\ c \end{bmatrix}$ lies in the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

Solution: The problem is equivalent to asking for which values of c there exist solutions to

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ c \end{bmatrix}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & c \end{array}\right]$$

We apply row operations to get to row echelon form

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & c \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -3 \\ 0 & 6 & c-9 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & c-3 \end{array}\right]$$

This system is consistent if and only if the term $c-3$ to the right of the row of zeros is also zero. This means that the only value of c for which our vector belongs to the given span is $c = 3$.