Math 54
Fall 2017
Practice Final Exam
Exam date: 12/14/17
Time Limit: 170 Minutes

Name: ______________________

Student ID: ______________________

GSI or Section: ______________________

This exam contains 9 pages (including this cover page) and 10 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

This is a closed book exam. No notes or calculators are permitted.
We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

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1. (10 points) Consider the matrix

$$A_x = \begin{bmatrix} 1 & 1 & 2 \\ x & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) (5 points) Find all values of x such A_x is invertible.

(b) (5 points) Compute A_2^{-1}.

2. (10 points) Consider the matrices

$$A = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

(a) (2 points) Which of these matrices is orthogonally diagonalizable?

(b) (2 points) Find the eigenvalues of the orthogonally diagonalizable matrix from part (a)

(c) (6 points) Find an orthogonal matrix P consisting of eigenvectors of the orthogonally diagonalizable matrix from part (a).

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

   (a) (2 points) If σ is the largest singular value of an $m \times n$ matrix A, then $\|Av\| \leq \sigma\|v\|$ for all $v \in \mathbb{R}^n$ where $\|u\| = \sqrt{u \cdot u}$.

   (b) (2 points) If A and B are similar $n \times n$ matrices so that $A = PBP^{-1}$, then $Py(t)$ is a solution to $x'(t) = Ax(t)$ for any $y(t)$ such that $y'(t) = By(t)$.

   (c) (2 points) Only square matrices can be squared.

   (d) (2 points) If $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$ such that $\langle v, v \rangle = v \cdot v$ for all $v \in \mathbb{R}^n$ then $\langle u, w \rangle = u \cdot w$ for all $u, w \in \mathbb{R}^n$.

   (e) (2 points) The set of solutions to $y''' + by'' + cy' + dy = 0$ is always a vector space of dimension three.

4. (10 points) Consider the function on $\mathbb{R}^3$ given by

$$x \star y = \sum_{m=1}^{3} m x_m y_m$$

for $x, y \in \mathbb{R}^3$.

(a) (3 points) Show that this is an inner product on $\mathbb{R}^3$.

(b) (4 points) Find an orthogonal basis for $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ with respect to the $\star$ inner product.

(c) (3 points) Find the closest vector in V (from (b)) to $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ when the distance is computed with the $\star$ inner product.

5. (10 points) Suppose that U is an orthogonal $n \times n$ matrix that is orthogonally diagonalizable.

   (a) (4 points) Show that the only eigenvalues of U are ± 1.

   (b) (3 points) Show that if U has only 1 as an eigenvalue then U is the $n \times n$ identity matrix.

   (c) (3 points) Give an example of an orthogonal and orthogonally diagonalizable matrix U with entries other than ± 1.

6. (10 points) Suppose that y_1 and y_2 are solutions to $y'' + b(t)y' + c(t)y = 0$ and $W(t) = W[y_1, y_2](t)$ is their Wronskian.

   (a) (4 points) Using only the defintion of $W(t)$, show that $W(t)$ satisfies the equation $W'(t) = -b(t)W(t)$. Deduce from your calculation that if $W(0) \neq 0$ then $W(t) \neq 0$ for all t.

   (b) (2 points) Show that if $y_1(t) \neq 0$ then

$$\left(\frac{y_2}{y_1}\right)' = \frac{W}{(y_1)^2}$$

(c) (4 points) Given that $y_1(t) = e^{-t^2}$ is a solution to $y'' + 4ty' + (4t^2 + 2)y = 0$, find another linearly independent solution using parts (a) and (b).

7. (10 points) Consider the differential equation $y'''' - 6y''' + 14y'' - 14y' + 5y = 0$.

   (a) (6 points) Find the general solution of the above differential equation. (*Hint:* $r = 2 + i$ is a root of the characteristic polynomial.)

   (b) (4 points) Solve the initial value problem where $y(0) = 1, y'(0) = 3, y''(0) = 5$, and $y'''(0) = 5$.

8. (a) (4 points) Find a basis of solutions to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

(b) (6 points) Find a particular solution to

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} t^{-1}e^{2t} \\ 0 \end{bmatrix}$$

9. (10 points) Consider the heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}.$$

(a) (6 points) Find the solution defined for $0 \le x \le \pi$ and $t \ge 0$ that satisfies $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \sin(3x) - 5\sin(5x)$.

(b) (4 points) Suppose that $v(x,t)$ is the solution defined for $0 \le x \le \pi$ and $t \ge 0$ that satisfies $v(0,t) = 0$, $v(\pi,t) = 1$ and $v(x,0) = \sin(3x) - 5\sin(5x)$. Find $\lim\limits_{t\to\infty} v(x,t)$.

10. (10 points) Consider the function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Euler famously computed $\zeta(2) = \pi^2/6$ in 1738. Evaluate $\zeta(4)$ by computing the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$, given by $f(x) = \pi^4 - x^4$. You may assume Euler's result.

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