Math 54
Fall 2017
Exam 1
9/26/17
Time Limit: 80 Minutes

Name: ______________________

Student ID: ______________________

GSI or Section: ______________________

This exam contains 6 pages (including this cover page) and 7 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

> This is a closed book exam. No notes or calculators are permitted.
> We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

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1. (10 points) Consider the matrix $A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 8 & 3 \end{bmatrix}$.

   (a) (3 points) Compute the determinant of A and explain why A is invertible.

   Solution: We have

$$\begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 8 & 3 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 8 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(-2) + 3(-1) = -1$$

   by cofactor expansion along the first column.
   A is invertible since $\det(A) \neq 0$.

   (b) (4 points) Find A^{-1}.

   Solution: We proceed via row reduction.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 11 & 6 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 0 & 1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 3 & -11/2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -2 & 5 & -1 \\ 0 & 1 & 0 & -3 & 6 & -1 \\ 0 & 0 & 1/2 & 3 & -11/2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 1 \\ 0 & 1 & 0 & -3 & 6 & -1 \\ 0 & 0 & 1 & 6 & -11 & 2 \end{array}\right]$$

   Therefore, we see that

$$A^{-1} = \begin{bmatrix} 2 & -5 & 1 \\ -3 & 6 & -1 \\ 6 & -11 & 2 \end{bmatrix}.$$

(c) (3 points) Use your answer from the previous part to solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix}$.

Solution: We have

$$\mathbf{x} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

2. (a) (5 points) Find all values of s such that $\mathbf{b} = \begin{pmatrix} s \\ 2 \\ 2 \\ 1 \end{pmatrix}$ is in the span of $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 3 \\ 2 \end{pmatrix}$.

Solution: We want to determine when $\mathbf{b}$ is a linear combination of the given vectors. To solve, we check when the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, where the columns of A are the given vectors. We row reduce the augmented matrix $[A \mid \mathbf{b}]$:

$$\left[\begin{array}{cc|c} -1 & 3 & s \\ 0 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -1 & 3 & s \\ 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 6 & 2+s \\ 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & s-4 \\ 0 & 0 & 0 \end{array}\right]$$

When $s - 4 \neq 0$, $A\mathbf{x} = \mathbf{b}$ will be inconsistent, and otherwise we will have a unique solution. Thus, $\mathbf{b}$ is in the span of the two given vectors whenever $s = 4$.

(b) (5 points) Determine the values of s for which the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ corresponding to the matrix below is onto. Then, determine the values of s for which it is one-to-one.

$$\begin{bmatrix} -1 & 3 & s \\ 0 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution: T is never onto since $3 < 4$ so there cannot be a pivot in every row.

T is one-to-one whenever there is a pivot in every column. By the row reduction from the previous part, we see that this is the case whenever $s \neq 4$.

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point. An additional point will be awarded for a correct brief justification. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

(a) (2 points) The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the span of the columns of A.

Solution: True.
A solution to the equation is precisely a description of $\mathbf{b}$ as a linear combination of the columns of A weighted by the entries in $\mathbf{x}$.

(b) (2 points) If A is an $m \times n$ matrix and $n > m$, then the columns of A are linearly dependent.

Solution: True.
The columns of A are linearly independent when there is a pivot in every column. Since there are more columns than rows, this is not possible and the columns must be linearly dependent.

(c) (2 points) If A is an $n \times n$ matrix and $\det(A) = 0$, then the system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

Solution: False.
Since $\det(A) = 0$, A is not invertible, and hence the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is not one-to-one. Thus, $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(d) (2 points) Any two $n \times n$ elementary matrices commute because they correspond to row operations.

Solution: False.
This would imply that all invertible matrices commute, which is certainly not true. For a specific example of elementary matrices which do not commute, consider the following two elementary matrices.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(e) (2 points) If A and B are $n \times n$ matrices and AB is invertible, then A and B are invertible.

Solution: True.
Since $\det(AB) \neq 0$ and $\det(AB) = \det(A)\det(B)$, we see that $\det(A) \neq 0$ and $\det(B) \neq 0$.

4. (10 points) Find all values of a, b, c so that the vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ solves the linear system associated to the following augmented matrix.

$$\left[\begin{array}{ccc|c} a+1 & b & 0 & c \\ 0 & c & a & 2 \\ a+b & -1 & -c & 0 \end{array}\right]$$

Solution: The above matrix corresponds to the system of equations

$$\begin{aligned} (a+1)x_1 + bx_2 &= c \\ cx_2 + ax_3 &= 2 \\ (a+b)x_1 - x_2 - cx_3 &= 0 \end{aligned}$$

In order for $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ to be a solution we must have

$$\begin{aligned} a + 1 - b &= c \\ -c + 2a &= 2 \\ a + b + 1 - 2c &= 0 \end{aligned}$$

which is the same as

$$\begin{aligned} a - b - c &= -1 \\ 2a - c &= 2 \\ a + b - 2c &= -1 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & -2 & -1 \end{array}\right]$$

We row reduce it

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & -2 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

The third equation is $-2c = -4$ and so $c = 2$. Plugging that value into the second equation we get $2b + 2 = 4$ and so $b = 1$. Plugging back into the first equation we get $a - 1 - 2 = -1$ and so $a = 2$. Therefore, the only possible value for a, b, c is

$$a = 2, \ b = 1, \ c = 2$$

5. (10 points) For each of the following requirements, provide an example of an $n \times n$ matrix A satisfying those requirements **OR** explain why no such matrix exists. If you are providing an example, you can choose any value of n that you like.

   (a) (3 points) The columns of A span $\mathbb{R}^n$ and are linearly independent.

   Solution: A simple example is the identity matrix I_n for any n.

   (b) (3 points) The columns of A do not span $\mathbb{R}^n$, but are linearly independent.

   Solution: No such A exists. A would have to have a pivot in every column but not in every row. However, the number of rows and columns of A are the same.

   (c) (4 points) The columns of A do not span $\mathbb{R}^n$, and are not linearly independent.

   Solution: There are many examples. A simple one is the following 2×2 matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

6. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by multiplication by a 2×2 matrix A. Let S be a subset of $\mathbb{R}^2$.

   (a) (3 points) Write a formula relating the area of S and the area of $T(S)$.

   Solution:
   $$\text{Area}(T(S)) = |\det(A)|\text{Area}(S)$$

   (b) (4 points) Suppose that T is the linear transformation given by rotation $\pi/2$ radians counterclockwise around the origin, then reflection over the x-axis, and then scaling by 3 in the y-direction. Find A.

   Solution: Let's break down what T does to $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.
   $$\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix}0\\-1\end{pmatrix} \xrightarrow{\text{scale}} \begin{pmatrix}0\\-3\end{pmatrix}$$
   $$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix}-1\\0\end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix}-1\\0\end{pmatrix} \xrightarrow{\text{scale}} \begin{pmatrix}-1\\0\end{pmatrix}$$
   Therefore, we have
   $$A = \begin{bmatrix}0 & -1\\-3 & 0\end{bmatrix}$$

   (c) (3 points) If S is the interior of the circle of radius 2 (which has area 4π). Find the area of $T(S)$ where T is as in part (b).

   Solution: Applying the formula from part (a), we get
   $$\text{Area}(T(S)) = |\det(A)|\text{Area}(S) = 3(4\pi) = 12\pi.$$

7. (10 points) Consider the two equations below.

$$x_1 + 4x_2 - 5x_3 = 0$$

$$2x_1 - x_2 + 8x_3 = 9$$

(a) (3 points) For each equation, describe its solution set geometrically; for example, your answer could be of the form: a plane, a sphere, etc.

Solution: Each equation determines a plane in $\mathbb{R}^3$.

(b) (7 points) Determine if these two sets intersect and if so, describe their intersection geometrically as in part (a).

Solution: To check if the planes have an intersection, we need to check if the system is consistent. For that, we write the augmented matrix as:

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{array}\right]$$

We apply row operations to get to row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{array}\right] \xrightarrow{R_2 \to R_2/-9} \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

Now, the matrix is in row echelon form. We need to do one more step to get it in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

This is equivalent to the following system.

$$x_1 + 3x_3 = 4$$

$$x_2 - 2x_3 = -1$$

Thus,

$$x_1 = 4 - 3x_3$$

$$x_2 = -1 + 2x_3$$

with x_3 free, which can be written in parametric vector form as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} x_3$$

Therefore, the two planes intersect, and their intersection is a line.

Extra space.

Extra space.