It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

Name and section: ______________________________

1. (5 points) Compute the rank of $A = \begin{bmatrix} 1 & -3 & 5 & 3 \\ -3 & 4 & -6 & -8 \\ 0 & -5 & 9 & 1 \end{bmatrix}$. What can you conclude about the dimension of the null space of A?

Solution: The rank of a matrix is the number of pivots it has in echelon form, so we'll do row reduction. This gives

$$\begin{bmatrix} 1 & -3 & 5 & 3 \\ -3 & 4 & -6 & -8 \\ 0 & -5 & 9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 5 & 3 \\ 0 & -5 & 9 & 1 \\ 0 & -5 & 9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 5 & 3 \\ 0 & -5 & 9 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We get a matrix with 2 pivots, so the rank of A is $\boxed{2}$, and since we know that the rank and the dimension of the null space must sum up to the number of columns, which is 4, the dimension of the null space is $4 - 2 = \boxed{2}$. (Equivalently, we know that the dimension of the null space is the number of columns without pivots.)

2. (5 points) Compute the determinant of $B = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -3 & 4 & 5 \\ 0 & 2 & -1 & -1 \\ -4 & 9 & -4 & -8 \end{bmatrix}$. Is B invertible?

Solution: It doesn't look easy to compute this determinant using cofactor expansion, so instead we'll compute it using row reduction. It helps that we already have a pivot. This gives

$$\begin{vmatrix} 1 & -2 & 1 & 3 \\ 3 & -3 & 4 & 5 \\ 0 & 2 & -1 & -1 \\ -4 & 9 & -4 & -8 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 & 3 \\ 0 & 3 & 1 & -4 \\ 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 2 & -1 & -1 \\ 0 & 3 & 1 & -4 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & 1 & -16 \end{vmatrix}$$

At this point we can stop row reducing and just use cofactor expansion along the first two rows. This gives

$$-\begin{vmatrix} -1 & -9 \\ 1 & -16 \end{vmatrix} = -(16+9) = \boxed{-25}.$$

In particular, the determinant is not zero, so we conclude that B is invertible.