Name and section: ______________________________

1. (5 points) Give the general solution to the following equation:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

**Solution:** The eigenvalues of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are $3, -1$ and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for example. Then the general solution is

$$\mathbf{y}(t) = Ae^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. (5 points) Find all possible real values for $\lambda$ such that ODE $\lambda y'' + y = 0$ with boundary value $y(0) = 0, y(\pi/2) = 0$ has non-trivial solution.

**Solution:**Let's discuss three cases.

If $\lambda = 0$ then the ODE becomes $y = 0$ which obviously has only trivial solution. So this is not the case.

If $\lambda \neq 0$ then divide on both sides by $\lambda$, we have $y'' + (1/\lambda)y = 0$. The characteristic polynomial is $r^2 + 1/\lambda = 0$, so depending on the sign of $\lambda$ we have two more cases:

If $\lambda < 0$, then $-1/\lambda > 0$ and the roots will be $\sqrt{-1/\lambda}$ and $-\sqrt{-1/\lambda}$. Then the general solution is

$$y = Ae^{\sqrt{-1/\lambda}t} + Be^{-\sqrt{-1/\lambda}t}$$

Then $y(0) = A+B, y(\pi/2) = Ae^{\sqrt{-1/\lambda}\pi/2} + Be^{-\sqrt{-1/\lambda}\pi/2}$. The boundary value condition is now

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{-1/\lambda}\pi/2} & e^{-\sqrt{-1/\lambda}\pi/2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix of the above equation is non-zero because $\sqrt{-1/\lambda}\pi/2 \neq 0, \sqrt{-1/\lambda}\pi/2 \neq -\sqrt{-1/\lambda}\pi/2$. Thus we ruled out the case $\lambda < 0$.

If $\lambda > 0$, then the roots will be $\sqrt{1/\lambda}i$ and $-\sqrt{1/\lambda}i$. Then the general solution is

$$y = A\cos(\sqrt{1/\lambda}t) + B\sin(\sqrt{1/\lambda}t)$$

Now by boundary value condition, $y(0) = 0$, which means $A\cos(0) + B\sin(0) = A = 0$. By another boundary value condition $y(\pi/2) = 0$ where

$$y(\pi/2) = A\cos(\sqrt{1/\lambda}\pi/2) + B\sin(\sqrt{1/\lambda}\pi/2) = B\sin(\sqrt{1/\lambda}\pi/2)$$

since we already know that $A = 0$.

Now we can't let $B$ be 0, otherwise we will have trivial solution. Then $\sin(\sqrt{1/\lambda}\pi/2) = 0$. We know that for $\sin(x) = 0$, all possible solutions are $x = n\pi$ for integers $n$. Then $\sqrt{1/\lambda}\pi/2$ must be $n\pi$ for integers $n$. Then the possible values for $\lambda$ are $1/(4n^2)$ for all integers $n$.