MATH 54 – Linear Algebra and Differential Equations Quiz 11

Name: ____________________

You have 20 minutes to complete the quiz.

1. (5 points) Solve the differential equation $y'' - y = 0$ subject to the initial conditions $y(0) = 5, y'(0) = -1$.

Solution: The auxiliary polynomial is $r^2 - 1 = (r-1)(r+1)$, so there are two distinct real roots, $r_1 = 1$ and $r_2 = -1$. The general solution is $y(t) = c_1 e^t + c_2 e^{-t}$. Taking the derivative, we have $y'(t) = c_1 e^t - c_2 e^{-t}$. Then setting $t = 0$ gives $y(0) = c_1 + c_2 = 5$ and $y'(0) = c_1 - c_2 = -1$. This linear system has the solution $c_1 = 2, c_2 = 3$, so the solution to the differential equation is $\boxed{y(t) = 2e^t + 3e^{-t}}$.

2. (a) (2 points) Find the general solution to the homogeneous equation $y'' + 2y' + 2y = 0$. (Your final answer should only involve real-valued functions.)

Solution: The auxiliary equation is $r^2 + 2r + 2 = 0$, whose roots are $\frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$. The complex solutions to the differential equation are spanned by $e^{(-1+i)t}$ and $e^{(-1-i)t}$, but we want our functions to be real-valued, so we use $e^{-t}\cos t$ and $e^{-t}\sin t$ instead. (Recall: $e^{(-1+i)t} = e^{-t}e^{it} = e^{-t}(\cos t + i \sin t)$; we are using the real and imaginary parts of this function. We can similarly calculate $e^{(-1-i)t} = e^{-t}(\cos t - i\sin t)$.) Our general solution is $\boxed{y(t) = e^{-t}(c_1 \cos t + c_2 \sin t)}$.

(b) (2 points) Use the method of undetermined coefficients to find one solution to the inhomogeneous equation $y'' + 2y' + 2y = \cos t$.

Solution: Our trial solution is $y_p(t) = A\cos t + B \sin t$. (Note that if $\cos t$ were a solution to the homogeneous equation, we would have to multiply this function by t, as otherwise $y_p'' + 2y_p' + 2y_p = 0$ regardless of A and B.) To solve for A and B, we plug everything in:

$$\begin{aligned} y_p(t) &= A\cos t + B\sin t && (1)\\ y_p'(t) &= -A\sin t + B\cos t && (2)\\ y_p''(t) &= -A\cos t - B\sin t && (3)\\ y_p'' + 2y_p' + 2y_p &= (-A + 2B + 2A)\cos t + (-B - 2A + 2B)\sin t && (4)\\ &= (A + 2B)\cos t + (-2A + B)\sin t. && (5) \end{aligned}$$

So we must solve the system $A + 2B = 1, -2A + B = 0$. We could do this by row reducing, or we can simply observe that $B = 2A$, so $A + 2(2A) = 5A = 1$. This gives $A = 1/5$ and $B = 2/5$, so $\boxed{y_p(t) = \frac{\cos t + 2\sin t}{5}}$.

(c) (1 point) Find the general solution to the inhomogeneous equation $y'' + 2y' + 2y = \cos t$.

Solution: If T denotes the linear transformation $T(y) = y'' + 2y' + 2y$ (from the vector space of real-valued functions to itself), then we found the kernel of T in part (a), and we now know one solution to $T(y) = b = \cos t$. The general solution is given by adding an arbitrary element of the kernel to this particular solution: $y = \frac{\cos t + 2\sin t}{5} + e^{-t}(c_1 \cos t + c_2 \sin t)$.