1. (5 points) Solve the differential equation $y'' - y = 0$ subject to the initial conditions $y(0) = 5, y'(0) = -1$.

**Solution:** The auxiliary polynomial is $r^2 - 1 = (r - 1)(r + 1)$, so there are two distinct real roots, $r_1 = 1$ and $r_2 = -1$. The general solution is $y(t) = c_1 e^t + c_2 e^{-t}$. Taking the derivative, we have $y'(t) = c_1 e^t - c_2 e^{-t}$. Then setting $t = 0$ gives $y(0) = c_1 + c_2 = 5$ and $y'(0) = c_1 - c_2 = -1$. This linear system has the solution $c_1 = 2, c_2 = 3$, so the solution to the differential equation is $y(t) = 2e^t + 3e^{-t}$. 
2. (a) (2 points) Find the general solution to the homogeneous equation $y'' + 2y' + 2y = 0$. (Your final answer should only involve real-valued functions.)

**Solution:** The auxiliary equation is $r^2 + 2r + 2 = 0$, whose roots are $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. The complex solutions to the differential equation are spanned by $e^{(-1+i)t}$ and $e^{(-1-i)t}$, but we want our functions to be real-valued, so we use $e^{-t} \cos t$ and $e^{-t} \sin t$ instead. (Recall: $e^{(-1+i)t} = e^{-t} e^{it} = e^{-t} (\cos t + i \sin t)$; we are using the real and imaginary parts of this function. We can similarly calculate $e^{(-1-i)t} = e^{-t} (\cos t - i \sin t)$.) Our general solution is $y(t) = e^{-t} (c_1 \cos t + c_2 \sin t)$.

(b) (2 points) Use the method of undetermined coefficients to find one solution to the inhomogeneous equation $y'' + 2y' + 2y = \cos t$.

**Solution:** Our trial solution is $y_p(t) = A \cos t + B \sin t$. (Note that if $\cos t$ were a solution to the homogeneous equation, we would have to multiply this function by $t$, as otherwise $y_p'' + 2y_p' + 2y_p = 0$ regardless of $A$ and $B$.) To solve for $A$ and $B$, we plug everything in:

- $y_p(t) = A \cos t + B \sin t$  
- $y_p'(t) = -A \sin t + B \cos t$  
- $y_p''(t) = -A \cos t - B \sin t$  
- $y_p'' + 2y_p' + 2y_p = (-A + 2B + 2A) \cos t + (-B - 2A + 2B) \sin t$  

So we must solve the system $A + 2B = 1$, $-2A + B = 0$. We could do this by row reducing, or we can simply observe that $B = 2A$, so $A + 2(2A) = 5A = 1$. This gives $A = 1/5$ and $B = 2/5$, so $y_p(t) = \frac{\cos t + 2 \sin t}{5}$.

(c) (1 point) Find the general solution to the inhomogeneous equation $y'' + 2y' + 2y = \cos t$.

**Solution:** If $T$ denotes the linear transformation $T(y) = y'' + 2y' + 2y$ (from the vector space of real-valued functions to itself), then we found the kernel of $T$ in part (a), and we now know one solution to $T(y) = b = \cos t$. The general solution is given by adding an arbitrary element of the kernel to this particular solution: $y = \frac{\cos t + 2 \sin t}{5} + e^{-t} (c_1 \cos t + c_2 \sin t)$. 

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