Name: ______________________________

You have 20 minutes to complete the quiz.

1. Consider the following inner product on the vector space $\mathbb{P}_2$ of quadratic polynomials. We define:

$$\langle f(t), g(t)\rangle = \int_0^1 f(t)g(t)dt$$

Find a basis of $\mathbb{P}_2$ which is orthogonal with respect to this inner product.

Solution: We take the basis $\{1, t, t^2\}$ and apply Gram-Schmidt. Let our first vector be $v_1 = 1$. Then we project t onto $\text{Span}(v_1)$ as follows:

$$\begin{aligned} v_2 &= t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle}1 \\ &= t - \frac{1}{2} \end{aligned}$$

Now to find our final vector v_3, we want to project t^2 onto $\text{Span}(v_1, v_2)$ as follows:

$$\begin{aligned} v_2 &= t^2 - \frac{\langle t^2, t - \frac{1}{2}\rangle}{\langle t - \frac{1}{2}, t - \frac{1}{2}\rangle}(t - \frac{1}{2}) - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 \\ &= t^2 - \frac{\frac{1}{12}}{\frac{1}{12}}(t - \frac{1}{2}) - \frac{1}{3}1 \\ &= t^2 - t + \frac{1}{6} \end{aligned}$$

The basis $\{1, t - \frac{1}{2}, t^2 - t + \frac{1}{6}\}$ is orthogonal as desired.

2. Find the least squares solutions to the system:

$$x + y + z = 2$$
$$x + y + z = 4$$
$$x - y + z = 6$$

Solution: First, we show how to solve this explicitly, then present a shortcut. We are trying to solve $A\vec{x} = b$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. To do this, we must project b onto $\text{Col}(A)$. To do this, we must find an orthogonal basis for $\text{Col}(A)$. Clearly the first to columns of A form a basis for the column space, so we apply Gram-Schmidt to this set. Let $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Then we let v be the first element of our orthogonal basis, and we compute the second element as follows:

$$\begin{aligned} v' &= w - \frac{v \cdot w}{v \cdot v} v \\ &= \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} \end{aligned}$$

So now we project b onto $\text{Span}(v, v')$ as follows:

$$\begin{aligned} \hat{b} &= \frac{b \cdot v}{v \cdot v} v + \frac{b \cdot v'}{v' \cdot v'} v' \\ &= 4v - \frac{3}{2} v' \\ &= \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} \end{aligned}$$

So we must solve:

$$A\vec{x} = \hat{b}$$

By row reducing, we see that the solutions are given by $y = -\frac{3}{2}$, $z = \frac{9}{2} - x$ and x is free.

Alternatively, as is shown in the book, we could have just solved $A^T A\vec{x} = A^T b$, which reduces to:

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 12 \end{bmatrix}$$

which has the same solution set as above.