1. Provide an example of the following, or explain why no such example can exist:

   (a) Vectors $u, v \in \mathbb{R}^2$ with $u \cdot v = 3$ such that $\{u, v\}$ is also a basis for $\mathbb{R}^2$.

   Solution: Let $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$. Then we seek:

   $$u \cdot v = ac + bd = 3$$

   To ensure this is a basis, we also need:

   $$ad - bc \neq 0$$

   For example we can do $a = 2$, $b = c = d = 1$.

   (b) Vectors $u, v \in \mathbb{R}^3$ with $||u + v|| > ||u|| + ||v||$.

   Solution: This is impossible by the triangle inequality, which says $||u + v|| \leq ||u|| + ||v||$.

   (c) Vectors $u, v, w \in \mathbb{R}^3$ such that $\{u, v, w\}$ is an orthogonal set.

   Solution: Take $u = e_1$, $v = e_2$, $w = 0$.

2. Let A be an $n \times n$ matrix with real coefficients.

   (a) Show that A is not invertible if and only if 0 is an eigenvalue of A.

   Solution: 0 is an eigenvalue of $A \Leftrightarrow 0$ is a root of $\chi_A \Leftrightarrow \det(A - 0 \cdot \mathrm{Id}) = 0 \Leftrightarrow \det A = 0$.

   (b) Given that A has only one eigenvalue over $\mathbb{C}$ (with multiplicity n) and is diagonalisable show that A is diagonal.

   Solution: Suppose that $P^{-1}AP = \lambda \cdot \mathrm{Id}$ for λ the unique eigenvalue of A. Then $A = P(\lambda \cdot \mathrm{Id})P^{-1} = \lambda \cdot \mathrm{Id}$.

   (c) Conclude that

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   is not diagonalisable.

   Solution: $\det(A - z\,\mathrm{Id}) = (1-z)^3$ so the only eigenvalue of B is 1. However, B is not diagonal and so by (b) cannot be diagonalisable.

3. (10 points) Find a basis for the orthogonal complement of the image of the linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^4$ defined as following:

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} a_0 + a_1 + 2a_2 - a_3 \\ 2a_1 + 4a_2 - 2a_3 \\ -2a_0 \\ 0 \end{bmatrix}$$

Solution: The matrix for T relative to the basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$ is

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the image of T is the column space of the matrix above, say A. Note that the orthogonal complement is the null space of A^T. The RREF of A^T is

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives a basis

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

for $\text{Nul}(A^T) = \text{Col}(A)^\perp = \text{Im}(T)^\perp$.

4. Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Recall that the trace of A, denoted as $tr(A)$, is the sum of all the matrix entries on the diagonal of the matrix. Complete the following tasks:

   (a) Write out the characteristic polynomial of matrix A in terms of $tr(A)$ and $det(A)$.

   Solution:

$$\begin{aligned} det(A - \lambda I) &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= (a_{11}a_{22} - a_{12}a_{21}) - \lambda\ (a_{11} + a_{22}) + \lambda^2 \\ &= \lambda^2 - \lambda tr(A) + det(A) = 0 \end{aligned}$$

   (b) In order for the matrix A to have all-real eigenvalues, what must be true about $Tr(A)$ and $Det(A)$? Justify your answer.

   Solution: For there to be all-real eigenvalues, the characteristic equations, which is also a quadratic equation, must have real solution for the roots.

$$\lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4det(A)}}{2}$$

$$tr(A)^2 - 4det(A) \geq 0$$

$$det(A) \leq \left(\frac{tr(A)}{2}\right)^2$$

5. Below all matrices are $n \times n$ matrices with real coefficients. Mark the following as true or false.

(a) A must have an even number of non-real eigenvalues.

Solution: True, either with or without multiplicity. It's easier to explain why the answer is yes without multiplicity: if $\lambda = a + bi$ is an eigenvalue with $b \neq 0$ and complex eigenvector $v \in \mathbb{C}^n$, then its complex conjugate $\overline{\lambda} = a - bi$ must also be an eigenvalue, with eigenvector $\overline{v}$ (this means we take the complex conjugate of every entry of v). So the non-real eigenvalues come in conjugate pairs.

(b) If $v_1, v_2 \in \mathbb{R}^n$ are eigenvectors of A with different eigenvalues $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent.

Solution: True. If $c_1v_1 + c_2v_2 = 0$, then applying A gives

$$\begin{aligned} A(c_1v_1 + c_2v_2) &= c_1Av_1 + c_2Av_2 && (1)\\ &= c_1\lambda_1v_1 + c_2\lambda_2v_2 && (2)\\ &= 0. && (3) \end{aligned}$$

Subtracting $\lambda_1(c_1v_1+c_2v_2) = 0$ gives $c_2(\lambda_2-\lambda_1)v_2 = 0$, and since $v_2 \neq 0$ (being an eigenvector) and $\lambda_1 - \lambda_2 \neq 0$ (by assumption), we get $c_2 = 0$. This gives $c_1v_1 = 0$, and since $v_1 \neq 0$ this gives $c_1 = 0$.

(c) If $v_1, v_2 \in \mathbb{R}^n$ are eigenvectors of A with different eigenvalues $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are orthogonal.

Solution: False. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has two eigenvalues $1, 0$ with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which are not orthogonal, although they are linearly independent. More generally, specifying a pair of linearly independent vectors v_1, v_2 in $\mathbb{R}^2$ and a pair of distinct eigenvalues $\lambda_1 \neq \lambda_2$ for them uniquely specifies a matrix $A = PDP^{-1}$, where D is the diagonal matrix with entries λ_1, λ_2 and P is the matrix whose columns are v_1 and v_2. In this construction there's no reason for v_1 and v_2 to be orthogonal.
However, this is true if A is symmetric ($A = A^T$).

(d) The dimension of $\mathrm{Nul}(A)$ is the multiplicity of 0 as an eigenvalue of A.

Solution: False. The dimension of $\mathrm{Nul}(A)$ is at most the multiplicity of 0 as an eigenvalue of A, but can be less than it. For example, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has the property that $\dim \mathrm{Nul}(A) = 1$, with basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but it has characteristic polynomial λ^2, so the multiplicity of 0 as an eigenvalue is 2.
However, this is true if A is diagonalizable.

(e) The eigenvalues of AB are the product of the eigenvalues of A and B.

Solution: False. This statement should seem quite suspicious because the eigenvalues of a matrix don't come in any distinguished order, so there's no distinguished way to match up an eigenvalue of A with an eigenvalue of B to multiply them and get an eigenvalue of AB. For an explicit counterexample, take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

The eigenvalues of A and B are both just 0, but AB has eigenvalues both 0 and 1.

However, this is true if A and B are *simultaneously* diagonalizable: that is, there is a single matrix P such that $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$ where D_A, D_B are diagonal.

6. Let A be an $n \times n$ matrix with characteristic polynomial $-\lambda(\lambda-1)^2$. Explain whether or not the following can be true, and if it can, give an example:
   (a) $\text{Rank}(A) = 0$
   (b) $\text{Rank}(A) = 1$
   (c) $\text{Rank}(A) = 2$
   (d) $\text{Rank}(A) = 3$

Solution: The dimension of an eigenspace for an eigenvalue λ is always less than or equal to the multiplicity of λ in the characteristic polynomial. In this case, $\lambda = 0$ has multiplicity 1, so the $\lambda = 0$ eigenspace has dimension less than or equal to 1. However the $\lambda = 0$ eigenspace has to be at least one dimensional because $\lambda = 0$ is an eigenvalue, which means it has some nonzero eigenvector. So the $\lambda = 0$ eigenspace is exactly 1 dimensional. Since the $\lambda = 0$ eigenspace is the same as the null space, we see that $\text{Rank}(A) = 3 - 1 = 2$. Thus a), b) and d) are impossible.

To see that c) is possible, consider:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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7. Let $T : M_{2\times 2} \to M_{2\times 2}$ be the linear transformation given by

$$T(A) = A^T$$

where A^T is the transpose of A.

(a) Is T an isomorphism? If so, describe T^{-1}.

Solution: Yes. $T^{-1} = T$ since $(A^T)^T = A$.

(b) Find the eigenvalues of T and the dimensions of the eigenspaces.

Solution: This can be done by writing a matrix of A, but it can actually be done directly. Suppose we have

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, $a = \lambda a$, $c = \lambda b$, $b = \lambda c$, and $d = \lambda d$. If a or d is nonzero, these imply immediately that $\lambda = 1$. Otherwise, either c or b is not zero, then either $c = \lambda b = \lambda^2 c$ or $b = \lambda^2 b$ implies that $\lambda = \pm 1$. Thus, the eigenvalues of T are 1 and -1.

For $\lambda = 1$, we have must have $c = b$ and no other conditions. Thus, the eigenspace for $\lambda = 1$ is

$$\left\{\begin{pmatrix} a & b \\ b & d \end{pmatrix} : a, b, d \in \mathbb{R}\right\} = \text{Span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

and this eigenspace has dimension equal to 3.

For $\lambda = -1$, we must have $a = 0$ since $a = -a$ and similarly $d = 0$. We also have $b = -c$. Thus, the eigenspace is

$$\left\{\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbb{R}\right\} = \text{Span}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

and this eigenspace has dimension equal to 1.

(c) Is there a basis for $M_{2\times 2}$ such that the matrix of T is diagonal with respect to this basis?

Solution: Yes. The sum of the dimensions of the eigenspaces is

$$3 + 1 = 4 = \dim M_{2\times 2}$$

so there is a basis for which the matrix of T is diagonal with respect to that basis. Namely, combining the two bases listed in the solution of the previous part will give one such basis.