1. Provide an example of the following, or explain why no such example can exist:
   (a) Vectors $u, v \in \mathbb{R}^2$ with $u \cdot v = 3$ such that $\{u, v\}$ is also a basis for $\mathbb{R}^2$.

   **Solution:** Let $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$. Then we seek:
   
   $$u \cdot v = ac + bd = 3$$

   To ensure this is a basis, we also need:
   
   $$ad - bc \neq 0$$

   For example we can do $a = 2$, $b = c = d = 1$.

   (b) Vectors $u, v \in \mathbb{R}^3$ with $\|u + v\| > \|u\| + \|v\|$.

   **Solution:** This is impossible by the triangle inequality, which says $\|u + v\| \leq \|u\| + \|v\|$.

   (c) Vectors $u, v, w \in \mathbb{R}^3$ such that $\{u, v, w\}$ is an orthogonal set.

   **Solution:** Take $u = e_1$, $v = e_2$, $w = 0$. 
2. Let $A$ be an $n \times n$ matrix with real coefficients.

(a) Show that $A$ is not invertible if and only if 0 is an eigenvalue of $A$.

**Solution:** 0 is an eigenvalue of $A \iff 0$ is a root of $\chi_A \iff \det(A - 0 \cdot \text{Id}) = 0 \iff \det A = 0$.

(b) Given that $A$ has only one eigenvalue over $\mathbb{C}$ (with multiplicity $n$) and is diagonalisable show that $A$ is diagonal.

**Solution:** Suppose that $P^{-1}AP = \lambda \cdot \text{Id}$ for $\lambda$ the unique eigenvalue of $A$. Then $A = P(\lambda \cdot \text{Id})P^{-1} = \lambda \cdot \text{Id}$.

(c) Conclude that

$$B = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}$$

is not diagonalisable.

**Solution:** $\det(A - z \text{Id}) = (1 - z)^3$ so the only eigenvalue of $B$ is 1. However, $B$ is not diagonal and so by (b) cannot be diagonalisable.
3. (10 points) Find a basis for the orthogonal complement of the image of the linear transformation $T : \mathbb{P}_3 \rightarrow \mathbb{R}^4$ defined as following:

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} a_0 + a_1 + 2a_2 - a_3 \\ 2a_1 + 4a_2 - 2a_3 \\ -2a_0 \\ 0 \end{bmatrix}$$

**Solution:** The matrix for $T$ relative to the basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$ is

$$
\begin{bmatrix}
1 & 1 & 2 & -1 \\
0 & 2 & 4 & -2 \\
-2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

So the image of $T$ is the column space of the matrix above, say $A$. Note that the orthogonal complement is the null space of $A^T$. The RREF of $A^T$ is

$$
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

This gives a basis

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

for $\text{Nul}(A^T) = \text{Col}(A)^\perp = \text{Im}(T)^\perp$. 


4. Given a matrix \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \). Recall that the trace of \( A \), denoted as \( tr(A) \), is the sum of all the matrix entries on the diagonal of the matrix. Complete the following tasks:

(a) Write out the characteristic polynomial of matrix \( A \) in terms of \( tr(A) \) and \( det(A) \).

**Solution:**

\[
\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\
= (a_{11}a_{22} - a_{12}a_{21}) - \lambda (a_{11} + a_{22}) + \lambda^2 \\
= \lambda^2 - \lambda tr(A) + det(A) = 0
\]

(b) In order for the matrix \( A \) to have all-real eigenvalues, what must be true about \( Tr(A) \) and \( Det(A) \)? Justify your answer.

**Solution:** For there to be all-real eigenvalues, the characteristic equations, which is also a quadratic equation, must have real solution for the roots.

\[
\lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4det(A)}}{2} \\
tr(A)^2 - 4det(A) \geq 0 \\
det(A) \leq \left( \frac{tr(A)}{2} \right)^2
\]
5. Below all matrices are \( n \times n \) matrices with real coefficients. Mark the following as true or false.

(a) \( A \) must have an even number of non-real eigenvalues.

**Solution:** True, either with or without multiplicity. It’s easier to explain why the answer is yes without multiplicity: if \( \lambda = a + bi \) is an eigenvalue with \( b \neq 0 \) and complex eigenvector \( v \in \mathbb{C}^n \), then its complex conjugate \( \overline{\lambda} = a - bi \) must also be an eigenvalue, with eigenvector \( \overline{v} \) (this means we take the complex conjugate of every entry of \( v \)). So the non-real eigenvalues come in conjugate pairs.

(b) If \( v_1, v_2 \in \mathbb{R}^n \) are eigenvectors of \( A \) with different eigenvalues \( \lambda_1 \neq \lambda_2 \), then \( v_1 \) and \( v_2 \) are linearly independent.

**Solution:** True. If \( c_1 v_1 + c_2 v_2 = 0 \), then applying \( A \) gives

\[
A(c_1 v_1 + c_2 v_2) = c_1 Av_1 + c_2 Av_2
\]

\[
= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2
\]

\[
= 0.
\]

Subtracting \( \lambda_1 (c_1 v_1 + c_2 v_2) = 0 \) gives \( c_2 (\lambda_2 - \lambda_1) v_2 = 0 \), and since \( v_2 \neq 0 \) (being an eigenvector) and \( \lambda_1 - \lambda_2 \neq 0 \) (by assumption), we get \( c_2 = 0 \). This gives \( c_1 v_1 = 0 \), and since \( v_1 \neq 0 \) this gives \( c_1 = 0 \).

(c) If \( v_1, v_2 \in \mathbb{R}^n \) are eigenvectors of \( A \) with different eigenvalues \( \lambda_1 \neq \lambda_2 \), then \( v_1 \) and \( v_2 \) are orthogonal.

**Solution:** False. For example, \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \) has two eigenvalues 1, 0 with eigenvectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) which are not orthogonal, although they are linearly independent. More generally, specifying a pair of linearly independent vectors \( v_1, v_2 \) in \( \mathbb{R}^2 \) and a pair of distinct eigenvalues \( \lambda_1 \neq \lambda_2 \) for them uniquely specifies a matrix \( A = PDP^{-1} \), where \( D \) is the diagonal matrix with entries \( \lambda_1, \lambda_2 \) and \( P \) is the matrix whose columns are \( v_1 \) and \( v_2 \). In this construction there’s no reason for \( v_1 \) and \( v_2 \) to be orthogonal.

However, this is true if \( A \) is symmetric (\( A = A^T \)).

(d) The dimension of \( \text{Nul}(A) \) is the multiplicity of 0 as an eigenvalue of \( A \).

**Solution:** False. The dimension of \( \text{Nul}(A) \) is at most the multiplicity of 0 as an eigenvalue of \( A \), but can be less than it. For example, the matrix \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) has the property that \( \dim \text{Nul}(A) = 1 \), with basis \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), but it has characteristic polynomial \( \lambda^2 \), so the multiplicity of 0 as an eigenvalue is 2.

However, this is true if \( A \) is diagonalizable.

(e) The eigenvalues of \( AB \) are the product of the eigenvalues of \( A \) and \( B \).

**Solution:** False. This statement should seem quite suspicious because the eigenvalues of a matrix don’t come in any distinguished order, so there’s no distinguished way to match up an
eigenvalue of $A$ with an eigenvalue of $B$ to multiply them and get an eigenvalue of $AB$. For an explicit counterexample, take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (4)$$

The eigenvalues of $A$ and $B$ are both just 0, but $AB$ has eigenvalues both 0 and 1. However, this is true if $A$ and $B$ are simultaneously diagonalizable: that is, there is a single matrix $P$ such that $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$ where $D_A, D_B$ are diagonal.
6. Let $A$ be an $n \times n$ matrix with characteristic polynomial $-\lambda(\lambda-1)^2$. Explain whether or not the following can be true, and if it can, give an example:

- (a) Rank$(A) = 0$
- (b) Rank$(A) = 1$
- (c) Rank$(A) = 2$
- (d) Rank$(A) = 3$

**Solution:** The dimension of an eigenspace for an eigenvalue $\lambda$ is always less than or equal to the multiplicity of $\lambda$ in the characteristic polynomial. In this case, $\lambda = 0$ has multiplicity 1, so the $\lambda = 0$ eigenspace has dimension less than or equal to 1. However the $\lambda = 0$ eigenspace has to be at least one dimensional because $\lambda = 0$ is an eigenvalue, which means it has some nonzero eigenvector. So the $\lambda = 0$ eigenspace is exactly 1 dimensional. Since the $\lambda = 0$ eigenspace is the same as the null space, we see that Rank$(A) = 3 - 1 = 2$. Thus a), b) and d) are impossible.

To see that c) is possible, consider:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Let $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ be the linear transformation given by $T(A) = A^T$ where $A^T$ is the transpose of $A$.

(a) Is $T$ an isomorphism? If so, describe $T^{-1}$.

**Solution:** Yes. $T^{-1} = T$ since $(A^T)^T = A$.

(b) Find the eigenvalues of $T$ and the dimensions of the eigenspaces.

**Solution:** This can be done by writing a matrix of $A$, but it can actually be done directly. Suppose we have

\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} = T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then, $a = \lambda a$, $c = \lambda b$, $b = \lambda c$, and $d = \lambda d$. If $a$ or $d$ is nonzero, these imply immediately that $\lambda = 1$. Otherwise, either $c$ or $b$ is not zero, then either $c = \lambda b = \lambda^2 c$ or $b = \lambda^2 b$ implies that $\lambda = \pm 1$. Thus, the eigenvalues of $T$ are $1$ and $-1$.

For $\lambda = 1$, we have must have $c = b$ and no other conditions. Thus, the eigenspace for $\lambda = 1$ is

\[
\left\{ \begin{pmatrix} a \\ b \\ d \end{pmatrix} : a, b, d \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}
\]

and this eigenspace has dimension equal to 3.

For $\lambda = -1$, we must have $a = 0$ since $a = -a$ and similarly $d = 0$. We also have $b = -c$. Thus, the eigenspace is

\[
\left\{ \begin{pmatrix} 0 \\ b \\ -b \\ 0 \end{pmatrix} : b \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}
\]

and this eigenspace has dimension equal to 1.

(c) Is there a basis for $M_{2 \times 2}$ such that the matrix of $T$ is diagonal with respect to this basis?

**Solution:** Yes. The sum of the dimensions of the eigenspaces is

$3 + 1 = 4 = \dim M_{2 \times 2}$

so there is a basis for which the matrix of $T$ is diagonal with respect to that basis. Namely, combining the two bases listed in the solution of the previous part will give one such basis.