1. Provide an example of the following, or explain why no such example can exist:
   (a) Vectors $u, v \in \mathbb{R}^2$ with $u \cdot v = 3$ such that $\{u, v\}$ is also a basis for $\mathbb{R}^2$.
   (b) Vectors $u, v \in \mathbb{R}^3$ with $||u + v|| > ||u|| + ||v||$.
   (c) Vectors $u, v, w \in \mathbb{R}^3$ such that $\{u, v, w\}$ is an orthogonal set.

2. Let A be an $n \times n$ matrix with real coefficients.

   (a) Show that A is not invertible if and only if 0 is an eigenvalue of A.

   (b) Given that A has only one eigenvalue over $\mathbb{C}$ (with multiplicity n) and is diagonalisable show that A is diagonal.

   (c) Conclude that

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   is not diagonalisable.

3. (10 points) Find a basis for the orthogonal complement of the image of the linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^4$ defined as following:

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} a_0 + a_1 + 2a_2 - a_3 \\ 2a_1 + 4a_2 - 2a_3 \\ -2a_0 \\ 0 \end{bmatrix}$$

4. Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Recall that the trace of A, denoted as $tr(A)$, is the sum of all the matrix entries on the diagonal of the matrix. Complete the following tasks:

   (a) Write out the characteristic polynomial of matrix A in terms of $tr(A)$ and $det(A)$.

   (b) In order for the matrix A to have all-real eigenvalues, what must be true about $Tr(A)$ and $Det(A)$? Justify your answer.

5. Below all matrices are $n \times n$ matrices with real coefficients. Mark the following as true or false.

   (a) A must have an even number of non-real eigenvalues.

   (b) If $v_1, v_2 \in \mathbb{R}^n$ are eigenvectors of A with different eigenvalues $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent.

   (c) If $v_1, v_2 \in \mathbb{R}^n$ are eigenvectors of A with different eigenvalues $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are orthogonal.

   (d) The dimension of $\mathrm{Nul}(A)$ is the multiplicity of 0 as an eigenvalue of A.

   (e) The eigenvalues of AB are the product of the eigenvalues of A and B.

6. Let A be an $n \times n$ matrix with characteristic polynomial $-\lambda(\lambda - 1)^2$. Explain whether or not the following can be true, and if it can, give an example:
   (a) $\text{Rank}(A) = 0$
   (b) $\text{Rank}(A) = 1$
   (c) $\text{Rank}(A) = 2$
   (d) $\text{Rank}(A) = 3$

7. Let $T : M_{2\times 2} \to M_{2\times 2}$ be the linear transformation given by

$$T(A) = A^T$$

where A^T is the transpose of A.

(a) Is T an isomorphism? If so, describe T^{-1}.

(b) Find the eigenvalues of T and the dimensions of the eigenspaces.

(c) Is there a basis for $M_{2\times 2}$ such that the matrix of T is diagonal with respect to this basis?