1. For which values of a is the following matrix invertible?

$$\begin{pmatrix} a & 0 & 1 \\ -1 & a & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution: We compute the determinant and find when it is zero. Expanding across the top row gives:

$$\begin{vmatrix} a & 0 & 1 \\ -1 & a & 0 \\ 0 & 1 & 1 \end{vmatrix} = a \cdot (a \cdot 1 - 0 \cdot 1) + 1 \cdot (-1 \cdot 1 - a \cdot 0) = a^2 - 1$$

So the matrix is invertible unless $a = \pm 1$.

2. Label the following statements as either true or false.
   (a) $\det A^T = \det A$
   (b) A matrix A is invertible if there is another matrix B such that $AB = I$.
   (c) The dimension of a subspace of $\mathbb{R}^n$ is at most n.
   (d) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.
   (e) If A is a square matrix, then after adding 2 times the first row of A to the second row, the determinant is multiplied by 2.
   (f) Every subspace of $\mathbb{R}^n$ contains at most n vectors.
   (g) If a 3×5 matrix A represents a surjective linear transformation, then $\text{Null}(A)$ must be exactly 2-dimensional.
   (h) If A and B are $n \times n$ matrices and AB is invertible, then BA must be invertible too.

Solution: a) is true. For example, using row reduction you can express A as a product of elementary matrices, and then it suffces to verify that $\det E^T = \det E$ where E is an elementary matrix.

b) is false. Either we also need to require that $BA = I$, or we need to require that A and B are square. Otherwise, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ gives a counterexample.

c) is true. The dimension of a subspace of $\mathbb{R}^n$ can be computed as the number of pivots in any matrix whose columns span that subspace, and because such a matrix has n rows, it can have at most n pivots.

d) is false. The correct rule is that $(AB)^{-1} = B^{-1}A^{-1}$, because $ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$ and similarly for multiplication on the other side. Most pairs of invertible $n \times n$ matrices will give a counterexample, such as $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

e) is false. The determinant is unchanged. It would be multiplied by 2 if we just multiplied a row by 2.

f) is false: $\mathbb{R}^n$ itself is a subspace of $\mathbb{R}^n$, and contains infinitely many vectors (if $n > 0$).

g) is true. If the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ is surjective, then its rank is 3, so the null space has dimension $5 - 3 = 2$ by the rank theorem.

h) is true. A matrix is invertible if and only if its determinant is nonzero. But since $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$, if AB has nonzero determinant, so does BA.

3. A linear transformation, $T : \mathbb{R}^3 \to \mathbb{R}^3$, has the following effect:
$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\2\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\\1\end{bmatrix}, T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}.$
   (a) What is the standard matrix of the transformation?
   (b) Is the transformation one-to-one? Is it onto?
   (c) Find a basis for the column and null spaces.

Solution:

(a) Standard matrix $\begin{bmatrix}1 & 0 & 1\\-1 & -1 & 1\\2 & 1 & 0\end{bmatrix}$

(b) Row-reduce $\begin{bmatrix}1 & 0 & 1\\-1 & -1 & 1\\2 & 1 & 0\end{bmatrix}$ $[R_3 - 2R_1]R_2 + R_1$ $\begin{bmatrix}1 & 0 & 1\\0 & -1 & 2\\0 & 1 & -2\end{bmatrix}$ $[R_3 + R_2]R_2 \div -1$ $\begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$
This is neither one-to-one nor onto.

(c) The basis for the column space $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$ is formed by the linearly independent columns of A. Rank=2
Basis for the null space found by solving $A\mathbf{x} = \mathbf{0}$. x_3 is the free variable. Basis: $\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}$.
Dim(Nul(A))= 1

4. (a) Let A be a $n \times n$ matrix. Relate $\det(-A)$ to $\det(A)$. (Be careful)

Solution: We can pull out a scalar row by row, and since there are n rows we get $\det(-A) = (-1)^n \det(A)$

(b) Suppose A, B are $n \times n$ matrices. If AB is invertible show A and B must both be invertible.

Solution: $\det(AB) \neq 0$ and $\det(A)\det(B) = \det(AB)$ so $\det(A), \det(B) \neq 0$. Thus A, B are invertible.

(c) Suppose $A^k = 0$. Show that A cannot be invertible.

Solution: $\det(A^k) = \det(A)^k = 0$ so $\det(A) = 0$. Thus A cannot be invertible.

5. Let $\mathcal{B} = \{1, t-1, (t-1)^2\}$ be a subset of $\mathbb{P}_2$.

   (a) Show that $\mathcal{B}$ is a basis for $\mathbb{P}^2$.

   Solution: Under the usual isomorphism $\mathbb{P}_2 \to \mathbb{R}^3$, this amounts to show that the following determinant is non-zero:
   $$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$
   This is true because the determinant is 1.

   (b) Find the $\mathcal{B}$-coordinate of $1 + 2t + 3t^2$.

   Solution: By the Taylor expansion,
   $$1 + 2t + 3t^2 = 6 + 8(t-1) + 3(t-1)^2$$
   Therefore, the $\mathcal{B}$-coordinate is $(6, 8, 3)$.

6. Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at $(1,1,1), (2,3,4), (3,4,5)$, and $(4,5,7)$. Find its volume.

Solution: After translating $(1,1,1)$ to the origin, the other vertices are $(1,2,3), (2,3,4)$, and $(3,4,6)$. The volume of the parallelepiped with these vertices is the absolute value of the following determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{vmatrix} = -1$$

Therefore, the volume of S is $\frac{1}{6}$.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by rotating points $\frac{\pi}{4}$ radians counterclockwise around the origin, then reflecting them across the y axis. What is the standard matrix of T?

Solution: The columns of the standard matrix of a linear transformation T are given by applying T to the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. e_1 transforms as follows:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{rotate}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \xrightarrow{\text{reflect}} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \tag{1}$$

e_2 transforms as follows:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{rotate}} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \xrightarrow{\text{reflect}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \tag{2}$$

Hence the standard matrix of T is $\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

8. $\mathbb{R}[x]$ is the set of polynomials with real coefficients. It is a (real) vector space with the usual addition and scalar multiplication you know and love from high school. Differentiation, $\frac{\mathrm{d}}{\mathrm{d}x} : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator.

   (a) What is $\operatorname{Ker}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$?

   Solution:
   $$\operatorname{Ker}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \{\alpha \in \mathbb{R}\},$$
   that is, the constant polynomials, since the derivative of a polynomial is zero if and only if the polynomial is constant.

   (b) What is $\operatorname{Im}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$?

   Solution:
   $$\operatorname{Im}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \mathbb{R}[x].$$
   This is true because every polynomial is the derivative of its indefinite integral (where you choose any value for C. For example, $2x$ is in the image because it is the derivative of x^2, which is the indefinite integral of $2x$ with $C = 0$.

   (c) Is $\frac{\mathrm{d}}{\mathrm{d}x}$ injective? Is $\frac{\mathrm{d}}{\mathrm{d}x}$ surjective?

   Solution: It is not injective as it has nonzero kernel. It is surjective because the image is all of $\mathbb{R}[x]$.

   (d) Is $\dim(\mathbb{R}[x])$ finite? If so, what is it? If not, prove that it is not.

   Solution: It is not finite because we have exhibited a linear operator on $\mathbb{R}[x]$ which is surjective but not injective.