1. (10 points) Write the definition of each of the following concepts. Use complete sentences and be as precise as you can.

(a) (2 points) The inverse of a matrix.

Solution: The inverse of an \( n \times n \) matrix \( A \) is an \( n \times n \) matrix \( B \) so that \( AB = BA = I \), where \( I \) is the \( n \times n \) identity matrix.

(b) (2 points) The set of vectors \( \{v_1, \ldots, v_k\} \) in a vector space \( V \) being linearly independent.

Solution: This set of vectors is linearly independent if whenever:

\[
    c_1 v_1 + \ldots + c_k v_k = 0
\]

for some scalars \( c_1, \ldots, c_k \), then we must have \( c_1 = \ldots = c_k = 0 \).

(c) (2 points) The dimension of a (finite-dimensional) vector space. (State the theorems which make this definition meaningful.)

Solution: The dimension of a vector space is the size of any basis of the vector space. This is a meaningful definition because 1) every vector space has a basis, and 2) and two bases have the same size.

(d) (2 points) The projection of a vector \( v \) in \( \mathbb{R}^n \) onto a subspace \( W \).

Solution: It is the unique vector \( w \) in \( W \) such that \( v - w \in W^\perp \). Equivalently, it is the vector \( w \) minimizing \( \|v - w\| \).

(e) (2 points) A diagonalizable matrix.

Solution: A diagonalizable matrix is an \( n \times n \) matrix such that there exists another \( n \times n \) invertible matrix \( P \) such that \( A = PDP^{-1} \), where \( D \) is a diagonal matrix.
2. (10 points) Find the equation \( y = \alpha + \beta x \) of the least-squares line that best fits the data 

\[
(a_1, b_1) = (0, 1), (a_2, b_2) = (1, 1), (a_3, b_3) = (1, 2)
\]

That is, the equation minimizing \( \sum_{i=1}^{n=3} |b_i - (\alpha + \beta a_i)|^2 \).

**Solution:** This can be rewritten as the usual least square problem \( Ax = b \) where

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
\]

Since the columns are linearly independent, there is a unique solution for \( A^T Ax = A^T b \) which is

\[
\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}
\]
3. (10 points) A matrix is called *nilpotent* if $A^n = 0$ for some $n > 0$.

(a) (2 points) Write down an example of a nonzero nilpotent matrix.

**Solution:** The classic example is 
\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\]

(b) (4 points) Show that the only eigenvalue of a nilpotent matrix is zero.

**Solution:** Suppose $A^n = 0$. Then $Av = \lambda v \Rightarrow 0 = A^n v = \lambda^n v \Rightarrow \lambda^n = 0 \Rightarrow \lambda = 0$.

(c) (2 points) If $A$ is an $n \times n$ nilpotent matrix, what is the characteristic polynomial of $A$?

**Solution:** $\chi_A(z) = (-z)^n$, since the only eigenvalues are 0, so the only root can be 0. Also it has degree $n$, and therefore it must be $c \cdot z^n$. Since the leading term is always $(-1)^n$, we arrive at the answer.

(d) (2 points) Is $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ a nilpotent matrix?

**Solution:** One can calculate that 5 is an eigenvalue of $B$ and so $B$ is not nilpotent. Alternatively, any positive power of $A$ has only positive entries, and therefore can never be the zero matrix.
4. (10 points) Consider the following five functions.

1. \( \text{det} : M_{2 \times 2} \to \mathbb{R} \) given by taking a matrix \( A \) to its determinant \( \det(A) \).

2. \( T : M_{2 \times 2} \to M_{2 \times 2} \) given by \( T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A \).

3. \( e : M_{2 \times 2} \to M_{2 \times 2} \) given by \( e(A) = e^A \).

4. \( S : \mathbb{R}^7 \to \mathbb{R} \) given by \( S(v) = v \cdot w \) where \( w \in \mathbb{R}^7 \) is a fixed nonzero vector.

5. \( \Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \) given by \( \Delta(y(x)) = y''(x) \).

Here, \( M_{2 \times 2} \) is the space of 2 \( \times \) 2 real matrices, \( C^\infty(\mathbb{R}) \) is the space of infinitely differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \), and \( e^A \) is the matrix exponential.

(a) (3 points) Which of these maps are linear transformations?

\[ \textbf{Solution:} \] 2, 4, 5 are all easily seen to be linear. 1 is not linear as \( \det(cA) = c^2 \det(A) \neq c \cdot \det(A) \) in general and 2 is not linear as \( e^{cA} = e^c e^A \neq ce^A \) in general.

(b) (3 points) Of the maps that are linear, which are injective?

\[ \textbf{Solution:} \] Only 2. 2 is injective since it is an isomorphism (see solution to part d). 4 cannot be injective since \( 7 > 1 \) and \( \ker \Delta = \{ f(x) = Ax + B \} \) is 2-dimensional.

(c) (3 points) Of the maps that are linear, which are surjective?

\[ \textbf{Solution:} \] 2, 4, 5 are all surjective. 2 is injective since it is an isomorphism (see solution to part d). 4 is surjective since for any \( c \in \mathbb{R} \), we have that \( S\left( \frac{x}{w} w \right) = c \). 5 is surjective since given any \( y \in C^\infty(\mathbb{R}) \), we have that \( \Delta \left( \int_0^x \left( \int_0^t y(s) \, ds \right) \, dt \right) = y(x) \).

(d) (1 point) Of the maps that are linear, which are isomorphisms?

\[ \textbf{Solution:} \] Only 2. The other two are not injective so not isomorphisms. The inverse of 2 is given by \( A \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A \).
5. (10 points) Solve the following second order linear differential equation:

\[ y'' - 3y' + 2y = 10t \sin(t) - 4 \cos(t) \]

subject to the initial conditions:

\[ y(0) = 12, \quad y'(0) = 15 \]

**Solution:** First solve for the homogeneous equation:

\[ y'' - 3y' + 2y = 0 \]

Auxiliary Equation:

\[ r^2 - 3r + 2 = 0 \Rightarrow (r - 2)(r - 1) = 0 \Rightarrow r_1 = 1, \quad r_2 = 2 \]

\[ y_h = c_1 e^t + c_2 e^{2t} \]

Guess the form of the particular solution:

\[ y_p = (at + b) \sin(t) + (ct + d) \cos(t) \]

\[ y_p' = a \cos t + (at + b) \sin t + c \cos t - (ct + d) \sin t \]

\[ y_p'' = 2a \cos t - (at + b) \sin t - 2c \sin t - (ct + d) \cos t \]

\[ y_p'' - 3y_p' + 2y_p = (a + 3c) t \sin t + (c - 3a) t \cos t + (b + 3d - 3a - 2c) \sin t + (-3b + d + 2a - 3c) \cos t \]

By matching the coefficient of corresponding terms:

\[
\begin{bmatrix}
1 & 0 & 3 & 0 \\
-3 & 0 & 1 & 0 \\
-3 & 1 & 2 & 3 \\
2 & -3 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
=
\begin{bmatrix}
10 \\
0 \\
0 \\
-4
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
=
\begin{bmatrix}
1 \\
0 \\
3 \\
3
\end{bmatrix}
\]

\[ y_p = t \sin t + (3t + 3) \cos t \]

\[ y = y_h + y_p = c_1 e^t + c_2 e^{2t} + t \sin t + (3t + 3) \cos t \]

\[ y' = c_1 e^t + 2c_2 e^{2t} + t \cos t + (3t + 3) \sin t \]

Using Initial Condition at \( t = 0 \)

\[ y(0) = c_1 + c_2 + 3 = 12 \]

\[ y'(0) = c_1 + 2c_2 + 3 = 15 \]

\[ \Rightarrow c_1 = 6; c_2 = 3 \]

Final Solution:

\[ y = 6e^t + 3e^{2t} + t \sin t + (3t + 3) \cos t \]
6. (10 points) Give the general solution to the following differential equation system:

\[
y'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} y(t)
\]

We present three solutions because we are very kind.

**Solution:** Denote the coefficient matrix as \( A \). Then the general solution must be \( y = e^{At} C \) for a constant vector \( C \). The only problem remains is: what is \( e^{At} \)?

By definition,

\[
e^{At} = \sum_{n=0}^{+\infty} \frac{A^n t^n}{n!}
\]

so we should first calculate \( A^n \). For small numbers of \( n \), we have the following result:

\[
A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}
\]

Thus we may guess that

\[
A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}
\]

and this pattern indeed holds, as the following computation shows more explicitly:

\[
A^{n+1} = A^n \times A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}
\]

Using this result, now we have

\[
\sum_{n=0}^{+\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{+\infty} \frac{1}{n!} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} t^n
\]

\[
= \sum_{n=0}^{+\infty} \frac{1}{n!} \begin{bmatrix} t^n & nt^n \\ 0 & t^n \end{bmatrix}
\]

\[
= \sum_{n=0}^{+\infty} \frac{t^n/n!}{0} + \sum_{n=0}^{+\infty} \frac{nt^n/n!}{0} = \begin{bmatrix} \sum_{n=0}^{+\infty} t^n/n! & \sum_{n=0}^{+\infty} nt^n/n! \\ 0 & \sum_{n=0}^{+\infty} t^n/n! \end{bmatrix}
\]

Now \( \sum_{n=0}^{+\infty} \frac{t^n}{n!} = e^t \) by definition, and

\[
\sum_{n=0}^{+\infty} \frac{nt^n}{n!} = \sum_{n=1}^{+\infty} \frac{nt^n}{n!} = \sum_{n=1}^{+\infty} \frac{t^n}{(n-1)!}
\]

\[
= t \sum_{n=1}^{+\infty} \frac{t^{n-1}}{(n-1)!} = t \sum_{n=0}^{+\infty} \frac{t^n}{n!} = te^t
\]
Put the result back into the matrix, the solution is
\[ e^{At} C = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 te^t \\ c_2 e^t \end{bmatrix} \]

**Solution:** An alternative approach that uses matrix exponentials, but avoids messy computations is as follows. First, notice that:
\[ \begin{bmatrix} t & t \\ 0 & t \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \]

where
\[ \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} . \]

Call these matrices \( D \) and \( N \) respectively. Since they commute, we have \( e^{D+N} = e^D e^N \). Since:
\[ e^D = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \]

and
\[ e^N = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

we get:
\[ e^A = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \]

Now we proceed as in the end of the first solution.

**Solution:** An alternative solution that avoids matrix exponentials is as follows. Notice that our system is:
\[ \begin{align*}
  x' &= x + y \\
  y' &= y
\end{align*} \]

Solving the second tells us \( y = c_1 e^t \). Substituting into the top gives:
\[ x' - x = c_1 e^t \]

Solving this gives \( x = c_2 te^t + c_1 e^t \), as we got in the solution above.
7. (10 points)

(a) (6 points) Compute the Fourier series for the extension of

\[ f(x) = \begin{cases} 
0 & -\pi < x < 0 \\
1 & 0 \leq x \leq \pi 
\end{cases} \]

as a 2\pi periodic function.

**Solution:** We compute for \( n > 0 \)

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_0^\pi 1 \, dx = 1 \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \frac{1}{\pi} \int_0^\pi \cos(nx) \, dx = 0 \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \frac{1}{\pi} \int_0^\pi \sin(nx) \, dx = \frac{1 + (-1)^{n+1}}{n\pi} \]

from which we see for \( k > 0 \) that \( b_{2k} = 0 \)

and \( b_{2k-1} = \frac{2}{(2k-1)\pi} \).

Thus, the Fourier series for \( f(x) \) is

\[ F(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)x} \]

(b) (4 points) Use part (a) to find the sum of the convergent series

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \]

**Solution:** By the convergence theorem for Fourier series we have

\[ 1 = F\left(\frac{\pi}{2}\right) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} \]

so

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)} = -\frac{\pi}{4} \]
8. This problem concerns solutions to the heat equation
\[ \frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \]  
with periodic boundary conditions \( u(-L, t) = u(L, t), \frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t) \).

(a) (6 points) Using separation of variables \( u(x, t) = X(x)T(t) \) as usual, we end up having to solve
\[ X''(x) + \lambda X(x) = 0 \]  
with periodic boundary conditions \( X(-L) = X(L), X'(-L) = X'(L) \). For which \( \lambda \geq 0 \) does this boundary value problem have a nonzero solution, and what are those nonzero solutions?

**Solution:** When \( \lambda > 0 \), the general solution in this case is \( X(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x \). The boundary conditions give
\[ c_1 \cos \sqrt{\lambda}L - c_2 \sin \sqrt{\lambda}L = c_1 \cos \sqrt{\lambda}L + c_2 \sin \sqrt{\lambda}L \]  
\[ c_1 \sin \sqrt{\lambda}L + c_2 \cos \sqrt{\lambda}L = -c_1 \sin \sqrt{\lambda}L + c_2 \cos \sqrt{\lambda}L \]  
which gives \( c_2 \sin \sqrt{\lambda}L = c_1 \sin \sqrt{\lambda}L = 0 \). So nonzero solutions are possible iff \( \sin \sqrt{\lambda}L = 0 \), which is possible iff \( \sqrt{\lambda} = \frac{\pi n}{L} \) (for \( n \) a positive integer), so \( \lambda = \frac{\pi^2 n^2}{L^2} \). In this case the space of solutions is 2-dimensional, spanned by \( \cos \frac{\pi n x}{L}, \sin \frac{\pi n x}{L} \).

When \( \lambda = 0 \) the general solution is \( X(x) = c_1 + c_2 x \). The boundary conditions give \( c_1 - c_2 L = c_1 + c_2 L \) and \( c_2 = c_2 \), so in this case the space of solutions is 1-dimensional, spanned by the constant function 1.

(b) (4 points) What are the corresponding nonzero solutions to the heat equation?

**Solution:** The \( t \) part \( T(t) \) of the corresponding solutions to the heat equation satisfy \( T'(t) + \beta \lambda T(t) = 0 \), so \( T(t) \) is a scalar multiple of \( e^{-\beta \lambda t} \). This gives solutions
\[ e^{-\beta \frac{\pi^2 n^2 t}{L^2}} \cos \frac{\pi n x}{L}, e^{-\beta \frac{\pi^2 n^2 t}{L^2}} \sin \frac{\pi n x}{L} \]  
when \( \lambda > 0 \), and the constant solution 1 when \( \lambda = 0 \).
9. (10 points) The Laplace equation is an important class of Partial Differential Equations that are very prevalent in many physical problems (E&M, Fluid, Mechanics etc.) In this problem, we will examine the form of solutions to the Laplace Equation. The Laplace equation is given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(a) (4 points) First, suppose we can write the function $u(x, y)$ as $u(x, y) = X(x)Y(y)$. Based on this assumption, produce two separate ordinary differential equations for the functions $X$ and $Y$. (Hint: an unknown constant $k$ should be involved somewhere in your ODEs.)

**Solution:** Using separation of variables:

$$u(x, y) = X(x)Y(y)$$

Plug this form of solution back to the Laplace Equation given:

$$X''(x)Y(y) = -X(x)Y''(y)$$

$$\frac{X''}{X} = \frac{Y''}{Y} = k, \ k \text{ is some constant}$$

$$\Rightarrow \begin{cases} X''(x) + kX(x) = 0 \\ Y''(y) - kY(y) = 0 \end{cases}$$

(b) (6 points) Discuss the different situations when $k$ takes different values. Write out the general solutions for $X(x)$ and $Y(y)$. Based on the form of the solutions of $X$ and $Y$, Is it possible for a nontrivial solution $u(x, y)$ to be periodic with respect to both the $x$ and $y$ variables?

**Solution:** In any case of $k$, the auxiliary equation of $X$ equation has the form:

$$r^2 + k = 0 \Rightarrow r = \pm \sqrt{-k}$$

The auxiliary equation of $Y$ equation has the form:

$$r^2 - k = 0 \Rightarrow r = \pm \sqrt{k}$$

1. If $k = 0$
   $$X(x) = a_1 + a_2x$$
   $$Y(y) = b_1 + b_2y$$

2. If $k < 0, -k > 0$
   $$X(x) = a_1e^{\sqrt{-k}x} + a_2e^{-\sqrt{-k}x}$$
   $$Y(y) = b_1\cos(\sqrt{-k}y) + b_2\sin(\sqrt{-k}y)$$

3. If $k > 0, -k < 0$
   $$X(x) = a_1\cos(\sqrt{-k}x) + a_2\sin(\sqrt{-k}x)$$
   $$Y(y) = b_1e^{\sqrt{-k}y} + b_2e^{-\sqrt{-k}y}$$

Hence it is *not possible* for the solution to be periodic with respect to both variables. If $k = 0$, then $u(x, y)$ is linear in $x$ and $y$ and if it is nonzero it is not periodic in either direction. If $k > 0$, then it is periodic in the $x$-direction, but not the $y$-direction (as long as it isn’t the trivial solution, which is certainly periodic). The opposite is true when $k < 0$. 

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