1. (10 points) Write the definition of each of the following concepts. Use complete sentences and be as precise as you can.

   (a) (2 points) The inverse of a matrix.

   Solution: The inverse of an $n \times n$ matrix A is an $n \times n$ matrix B so that $AB = BA = I$, where I is the $n \times n$ identity matrix.

   (b) (2 points) The set of vectors $\{v_1, \dots v_k\}$ in a vector space V being *linearly independent*.

   Solution: This set of vectors is linearly independent if whenever:

   $$c_1 v_1 + \dots c_k v_k = 0$$

   for some scalars $c_1, \dots c_k$, then we must have $c_1 = \dots c_k = 0$.

   (c) (2 points) The dimension of a (finite-dimensional) vector space. (State the theorems which make this definition meaningful.)

   Solution: The dimension of a vector space is the size of any basis of the vector space. This is a meaningful definition because 1) every vector space has a basis, and 2) and two bases have the same size.

   (d) (2 points) The projection of a vector v in $\mathbb{R}^n$ onto a subspace W.

   Solution: It is the unique vector w in W such that $v - w \in W^{\perp}$. Equivalently, it is the vector w minimizing $||v - w||$.

   (e) (2 points) A diagonalizable matrix.

   Solution: A diagonalizable matrix is an $n \times n$ matrix such that there exists another $n \times n$ invertible matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix.

2. (10 points) Find the equation $y = \alpha + \beta x$ of the least-squares line that best fits the data

$$(a_1, b_1) = (0, 1), (a_2, b_2) = (1, 1), (a_3, b_3) = (1, 2)$$

That is, the equation minimizing $\sum_{n=1}^{n=3} |b_i - (\alpha + \beta a_i)|^2$.

Solution: This can be rewritten as the usual least square problem $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Since the columns are linearly independent, there is a unique solution for $A^T Ax = A^T b$ which is

$$\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

3. (10 points) A matrix is called *nilpotent* if $A^n = 0$ for some $n > 0$.

(a) (2 points) Write down an example of a nonzero nilpotent matrix.

Solution: The classic example is
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) (4 points) Show that the only eigenvalue of a nilpotent matrix is zero.

Solution: Suppose $A^n = 0$. Then $Av = \lambda v \Rightarrow 0 = A^n v = \lambda^n v \Rightarrow \lambda^n = 0 \Rightarrow \lambda = 0$.

(c) (2 points) If A is an $n \times n$ nilpotent matrix, what is the characteristic polynomial of A?

Solution: $\chi_A(z) = (-z)^n$, since the only eigenvalues are 0, so the only root can be 0. Also it has degree n, and therefore it must be $c \cdot z^n$. Since the leading term is always $(-1)^n$, we arrive at the answer.

(d) (2 points) Is $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ a nilpotent matrix?

Solution: One can calculate that 5 is an eigenvalue of B and so B is not nilpotent. Alternatively, any positive power of A has only positive entries, and therefore can never be the zero matrix.

4. (10 points) Consider the following five functions.

   1. $\det : M_{2\times 2} \to \mathbb{R}$ given by taking a matrix A to its determinant $\det(A)$.
   2. $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$.
   3. $e : M_{2\times 2} \to M_{2\times 2}$ given by $e(A) = e^A$.
   4. $S : \mathbb{R}^7 \to \mathbb{R}$ given by $S(v) = v \cdot w$ where $w \in \mathbb{R}^7$ is a fixed nonzero vector.
   5. $\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $\Delta(y(x)) = y''(x)$.

   Here, $M_{2\times 2}$ is the space of 2×2 real matrices, $C^\infty(\mathbb{R})$ is the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and e^A is the matrix exponential.

   (a) (3 points) Which of these maps are linear transformations?

   Solution: 2, 4, 5 are all easily seen to be linear. 1 is not linear as $\det(cA) = c^2 \det(A) \neq c \det(A)$ in general and 2 is not linear as $e^{cA} = e^c e^A \neq ce^A$ in general.

   (b) (3 points) Of the maps that are linear, which are injective?

   Solution: Only 2. 2 is injective since it is an isomorphism (see solution to part d). S cannot be injective since $7 > 1$ and $\operatorname{Ker} \Delta = \{f(x) = Ax + B\}$ is 2-dimensional.

   (c) (3 points) Of the maps that are linear, which are surjective?

   Solution: 2, 4, 5 are all surjective. 2 is injective since it is an isomorphism (see solution to part d). 4 is surjective since for any $c \in \mathbb{R}$, we have that $S\left(\frac{c}{w \cdot w} w\right) = c$. 5 is surjective since given any $y \in C^\infty(\mathbb{R})$, we have that $\Delta\left(\int_0^x \left(\int_0^t y(s)\ ds\right)\ dt\right) = y(x)$.

   (d) (1 point) Of the maps that are linear, which are isomorphisms?

   Solution: Only 2. The other two are not injective so not isomorphisms. The inverse of 2 is given by $A \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$.

5. (10 points) Solve the following second order linear differential equation:

$$y'' - 3y' + 2y = 10t\sin(t) - 4\cos(t)$$

subject to the initial conditions:

$$y(0) = 12, \quad y'(0) = 15$$

Solution: First solve for the homogeneous equation:

$$y'' - 3y' + 2y = 0$$

Auxiliary Equation:

$$r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow r_1 = 1,\ r_2 = 2$$

$$y_h = c_1e^t + c_2e^{2t}$$

Guess the form of the particular solution: $y_p = (at+b)\sin(t) + (ct+d)\cos(t)$

$$y_p' = a\text{sin}t + (at+b)\text{cos}t + c\text{cos}t - (ct+d)\text{sin}t$$

$$y_p'' = 2a\text{cos}t - (at+b)\text{sin}t - 2c\text{sin}t - (ct+d)\text{cos}t$$

$$y_p'' - 3y_p' + 2y_p = (a+3c)t\text{sin}t + (c-3a)t\text{cos}t + (b+3d-3a-2c)\text{sin}t + (-3b+d+2a-3c)\text{cos}t$$

By matching the coefficient of corresponding terms:

$$\begin{cases} a + 3c = 10 \\ -3a + c = 0 \\ -3a + b - 2c + 3d = 0 \\ 2a - 3b - 3c + d = -4 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -3 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & -3 & -3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow y_p = t\text{sin}t + (3t+3)\text{cos}t$$

$$y = y_h + y_p = c_1e^t + c_2e^{2t} + t\text{sin}t + (3t+3)\text{cos}t$$

$$y' = c_1e^t + 2c_2e^{2t} + \text{sin}t + t\text{cos}t + 3\text{cos}t - (3t+3)\text{sin}t$$

Using Initial Condition at $t = 0$

$$y(0) = c_1 + c_2 + 3 = 12$$

$$y(0)' = c_1 + 2c_2 + 3 = 15$$

$$\Rightarrow c_1 = 6; c_2 = 3$$

Final Solution:

$$\boxed{y = 6e^t + 3e^{2t} + t\text{sin}t + (3t+3)\text{cos}t}$$

6. (10 points) Give the general solution to the following differential equation system:

$$\mathbf{y}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{y}(t)$$

We present three solutions because we are very kind.

Solution: .Denote the coefficient matrix as A. Then the general solution must be $\mathbf{y} = e^{At}C$ for a constant vector C. The only problem remains is: what is e^{At}?

By definition,

$$e^{At} = \sum_{n=0}^{+\infty} \frac{A^n t^n}{n!}$$

so we should first calculate A^n. For small numbers of n, we have the following result:

$$A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Thus we may guess that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

and this pattern indeed holds, as the following computation shows more explicitly:

$$A^{n+1} = A^n \times A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

Using this result, now we have

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{A^n t^n}{n!} &= \sum_{n=0}^{+\infty} \frac{1}{n!} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} t^n \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!} \begin{bmatrix} t^n & nt^n \\ 0 & t^n \end{bmatrix} \\ &= \sum_{n=0}^{+\infty} \begin{bmatrix} t^n/n! & nt^n/n! \\ 0 & t^n/n! \end{bmatrix} \\ &= \begin{bmatrix} \sum_{n=0}^{+\infty} t^n/n! & \sum_{n=0}^{+\infty} nt^n/n! \\ 0 & \sum_{n=0}^{+\infty} t^n/n! \end{bmatrix} \end{aligned}$$

Now $\sum_{n=0}^{+\infty} \frac{t^n}{n!} = e^t$ by definition, and

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{nt^n}{n!} &= \sum_{n=1}^{+\infty} \frac{nt^n}{n!} \\ &= \sum_{n=1}^{+\infty} \frac{t^n}{(n-1)!} \\ &= t \sum_{n=1}^{+\infty} \frac{t^{n-1}}{(n-1)!} \\ &= t \sum_{n=0}^{+\infty} \frac{t^n}{n!} \\ &= te^t \end{aligned}$$

Put the result back into the matrix, the solution is

$$e^{At}C = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^t + c_2te^t \\ c_2e^t \end{bmatrix}$$

Solution: An alternative approach that uses matrix exponentials, but avoids messy computations is as follows. First, notice that:

$$\begin{bmatrix} t & t \\ 0 & t \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

where

$$\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}.$$

Call these matrices D and N respectively. Since they commute, we have $e^{D+N} = e^D e^N$. Since:

$$e^D = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$$

and

$$e^N = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

we get:

$$e^A = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

Now we proceed as in the end of the first solution.

Solution: An alternative solution that avoids matrix exponentials is as follows. Notice that our system is:

$$x' = x + y$$
$$y' = y$$

Solving the second tells us $y = c_1e^t$. Substituting into the top gives:

$$x' - x = c_1e^t$$

Solving this gives $x = c_2te^t + c_1e^t$, as we got in the solution above.

7. (10 points)

(a) (6 points) Compute the Fourier series for the extension of

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

as a 2π periodic function.

Solution: We compute for $n > 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{\pi} \int_0^{\pi} 1\ dx = 1$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx = \frac{1}{\pi} \int_0^{\pi} \cos(nx)\ dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ dx = \frac{1}{\pi} \int_0^{\pi} \sin(nx)\ dx = \frac{1 + (-1)^{n+1}}{n\pi}$$

from which we see for $k > 0$ that

$$b_{2k} = 0$$

and

$$b_{2k-1} = \frac{2}{(2k-1)\pi}.$$

Thus, the Fourier series for $f(x)$ is

$$F(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)}$$

(b) (4 points) Use part (a) to find the sum of the convergent series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}.$$

Solution: By the convergence theorem for Fourier series we have

$$1 = F\left(\frac{\pi}{2}\right) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)}$$

so

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)} = -\frac{\pi}{4}$$

8. This problem concerns solutions to the heat equation

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with *periodic* boundary conditions $u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$.

(a) (6 points) Using separation of variables $u(x,t) = X(x)T(t)$ as usual, we end up having to solve

$$X''(x) + \lambda X(x) = 0 \tag{2}$$

with periodic boundary conditions $X(-L) = X(L), X'(-L) = X'(L)$. For which $\lambda \geq 0$ does this boundary value problem have a nonzero solution, and what are those nonzero solutions?

Solution: When $\lambda > 0$, the general solution in this case is $X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. The boundary conditions give

$$c_1 \cos\sqrt{\lambda}L - c_2 \sin\sqrt{\lambda}L = c_1 \cos\sqrt{\lambda}L + c_2 \sin\sqrt{\lambda}L \tag{3}$$

$$c_1 \sin\sqrt{\lambda}L + c_2 \cos\sqrt{\lambda}L = -c_1 \sin\sqrt{\lambda}L + c_2 \cos\sqrt{\lambda}L \tag{4}$$

which gives $c_2 \sin\sqrt{\lambda}L = c_1 \sin\sqrt{\lambda}L = 0$. So nonzero solutions are possible iff $\sin\sqrt{\lambda}L = 0$, which is possible iff $\sqrt{\lambda} = \frac{\pi n}{L}$ (for n a positive integer), so $\lambda = \frac{\pi^2 n^2}{L^2}$. In this case the space of solutions is 2-dimensional, spanned by

$$\cos\frac{\pi n x}{L}, \sin\frac{\pi n x}{L}. \tag{5}$$

When $\lambda = 0$ the general solution is $X(x) = c_1 + c_2 x$. The boundary conditions give $c_1 - c_2 L = c_1 + c_2 L$ and $c_2 = c_2$, so in this case the space of solutions is 1-dimensional, spanned by the constant function 1.

(b) (4 points) What are the corresponding nonzero solutions to the heat equation?

Solution: The t part $T(t)$ of the corresponding solutions to the heat equation satisfy $T'(t) + \beta\lambda T(t) = 0$, so $T(t)$ is a scalar multiple of $e^{-\beta\lambda t}$. This gives solutions

$$e^{-\beta\frac{\pi^2 n^2 t}{L^2}} \cos\frac{\pi n x}{L}, e^{-\beta\frac{\pi^2 n^2 t}{L^2}} \sin\frac{\pi n x}{L} \tag{6}$$

when $\lambda > 0$, and the constant solution 1 when $\lambda = 0$.

9. (10 points) The Laplace equation is an important class of Partial Differential Equations that are very prevalent in many physical problems (E&M, Fluid, Mechanics etc.) In this problem, we will examine the form of solutions to the Laplace Equation. The Laplace equation is given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(a) (4 points) First, suppose we can write the function $u(x, y)$ as $u(x, y) = X(x)Y(y)$. Based on this assumption, produce two separate ordinary differential equations for the functions X and Y. (Hint: an unknown constant k should be involved somewhere in your ODEs.)

Solution: Using separation of variables:

$$u(x, y) = X(x)Y(y)$$

Plug this form of solution back to the Laplace Equation given:

$$X''(x)Y(y) = -X(x)Y''(y)$$
$$\frac{X''}{-X} = \frac{Y''}{Y} = k, k \text{ is some constant}$$
$$\Rightarrow \begin{cases} X''(x) + kX(x) = 0 \\ Y''(y) - kY(y) = 0 \end{cases}$$

(b) (6 points) Discuss the different situations when k takes different values. Write out the general solutions for $X(x)$ and $Y(y)$. Based on the form of the solutions of X and Y, Is it possible for a nontrivial solution $u(x, y)$ to be periodic with respect to both the x and y variables?

Solution: In any case of k, the auxilary equation of X equation has the form:

$$r^2 + k = 0 \Rightarrow r = \pm\sqrt{-k}$$

The auxilary equation of Y equation has the form:

$$r^2 - k = 0 \Rightarrow r = \pm\sqrt{k}$$

1. If $k = 0$

$$X(x) = a_1 + a_2 x$$
$$Y(y) = b_1 + b_2 y$$

2. If $k < 0, -k > 0$

$$X(x) = a_1 e^{\sqrt{-k}x} + a_2 e^{-\sqrt{-k}x}$$
$$Y(y) = b_1 cos(\sqrt{-k}y) + b_2 sin(\sqrt{-k}y)$$

3. If $k > 0, -k < 0$

$$X(x) = a_1 cos(\sqrt{-k}x) + a_2 sin(\sqrt{-k}x)$$
$$Y(y) = b_1 e^{\sqrt{-k}y} + b_2 e^{-\sqrt{-k}y}$$

Hence it is *not possible* for the solution to be periodic with respect to both variables. If $k = 0$, then $u(x, y)$ is linear in x and y and if it is nonzero it is not periodic in either direction. If $k > 0$, then it is periodic in the x-direction, but not the y-direction (as long as it isn't the trivial solution, which is certainly periodic). The opposite is true when $k < 0$.