# Welcome to Lecture 9

## Abstract Vector Spaces and Linear Transformations

"Mathematics is the art of giving the same name to different things."
Henri Poincaré

**Friday:** Quiz through §3.3

**Next week:** review session during Office Hours Thurs 2-4pm and extra Office Hours

**Tuesday 10/6:** Midterm 1 through §4.4

Observe: the following properties of $\mathbb{R}^n$ are all we ever use

1) addition: $\underline{u}, \underline{v} \rightsquigarrow \underline{u} + \underline{v}$
2) add. is associative
3) add. is commutative
4) zero vector $\underline{0}$ such that $\underline{u} + \underline{0} = \underline{u} = \underline{0} + \underline{u}$
5) negatives for add. exist
$\underline{u} \rightsquigarrow -\underline{u}$ such that $\underline{u} + (-\underline{u}) = \underline{0}$

6) <u>scaling</u>: $\underline{u}, c \rightsquigarrow c \cdot \underline{u}$, $c$ number

7) scaling is <u>associative</u>

8) scaling by <u>1 does nothing</u> $1 \cdot \underline{u} = \underline{u}$

9) <u>distributivity</u> in addition of <u>scalars</u>

$$(c+d) \cdot \underline{u} = c \cdot \underline{u} + d \cdot \underline{u}$$

10) <u>distributivity</u> in addition of <u>vectors</u>

$$c \cdot (\underline{u} + \underline{v}) = c \cdot \underline{u} + c \cdot \underline{v}$$

<u>Def</u> A <u>vector space</u> $V$ is a set (elements are called <u>vectors</u>) with two operations:

1) <u>addition</u>: $\underline{u}, \underline{v} \in V \rightsquigarrow \underline{u}+\underline{v} \in V$
2) <u>scaling</u>: $\underline{u} \in V \rightsquigarrow c \cdot \underline{u} \in V$

$c$ any number

satisfying above <u>properties 1)-10)</u>

<u>Examples:</u>

1) $V = \mathbb{R}^n$, vectors $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

2) $V = \{ f : \mathbb{R} \to \mathbb{R} \text{ function} \}$

examples of vectors

$$f(x) = x^2 - 2x + 3$$

$$f(x) = \cos(7x)$$

$$\vdots$$

addition: $(f+g)(x) = f(x) + g(x)$

scaling: $(cf)(x) = cf(x)$

<u>Exer</u>: Check properties 1)-10) hold

3) a) $V = \{ f : \mathbb{R} \to \mathbb{R}$ <u>even</u> fns $\}$

$f(-x) = f(x)$

b) $V = \{ f : \mathbb{R} \to \mathbb{R}$ <u>odd</u> fns $\}$

$f(-x) = -f(x)$

<u>Exer</u>: Check properties 1)-10) hold

4) $V = P = \{$ <u>poly. fns</u> $f : \mathbb{R} \to \mathbb{R} \}$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

specific examples

$f(x) = 2 - x^2 + 3x^{17}$

$f(x) = 0$ ← <u>zero</u> <u>vector</u>

$f(x) = x + x^2$

$\vdots$

5) $V = P_n = \{$ poly fns $f : \mathbb{R} \to \mathbb{R}$ of degree $\leq n \}$

(__Reminder__: degree of poly is biggest number $k$ so that $a_k \neq 0$)

specific examples for __$n=3$__

$$f(x) = 2 - x^2 + 3x^3$$

$$f(x) = 7x^2$$

$$f(x) = 6$$

$$\vdots$$

6) $V = S = \{ (a_1, a_2, a_3, \ldots) \}$

infinite sequences of numbers

specific examples of vectors

$(1, 0, 1, 0, 1, 2, 3, -8, \ldots)$

zero vector $\rightarrow$ $(0, 0, 0, 0, 0, \ldots)$

$(0, 0, 3, 0, 0, -2, \ldots)$

7) $V = M_{m \times n} = \{ m \times n \text{ matrices } A \}$

specific examples of vectors for

$\underline{m=3,\ n=2}$

$$\begin{bmatrix} 1 & 2 \\ -3 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \ldots$$

<u>zero</u> <u>vector</u>

**Caution** many tricky **non-examples**

1) $\{$ poly fns $f : \mathbb{R} \to \mathbb{R}$ of degree $= n$ $\}$

**For $n > 0$:** there is no zero vector!

(**Convention**: $f(x) = 0$ has degree $0$)

2) $\{$ $m \times n$ matrices in REF $\}$

addition is not defined: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ not REF

Reminder of key concepts for $\mathbb{R}^n$ that make sense for <u>any</u> <u>vector space</u> $V$

1) linear combinations
2) spans
3) linear independence
4) subspaces
5) bases
6) coordinates

<u>Exer</u> Let $V = P_3 = \{$ poly fns $f : \mathbb{R} \to \mathbb{R}$ of deg $\leq 3 \}$

Is the vector $1 + x - x^3$ in $\text{span}\{1 + x^3, 1 - 2x, x^2 - x^3\}$ ?

<u>Soln</u>: Want to solve

$$\begin{pmatrix} 1 \\ + 1 \cdot x \\ + 0 \cdot x^2 \\ - 1 \cdot x^3 \end{pmatrix} = a \begin{bmatrix} 1 \\ + 0 \cdot x \\ + 0 \cdot x^2 \\ + 1 \cdot x^3 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -2x \\ + 0x^2 \\ + 0x^3 \end{bmatrix} + a_3 \begin{pmatrix} 0 \\ + 0x \\ + 1 \cdot x^2 \\ - 1 \cdot x^3 \end{pmatrix}$$

Equivalent to solving

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -\frac{5}{2} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{5}{2} \end{array}\right]$$

Pivot in augmentation column

$\Longrightarrow$ no solns

<u>Conclusion</u> vector is not in span.

<u>Exer</u> Let $V = M_{2\times 2} = \{2\times 2 \text{ matrices}\}$

Are the following vectors lin indep

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$A_4 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

<u>Soln</u>: Convert $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

into col vector $\underline{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Equivalent to question of lin indep of cols of $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no pivot in last col so <u>not</u> lin indep

<u>Exer</u> Let $V = P_2 = \{$ poly fns $f : \mathbb{R} \to \mathbb{R}$ of deg $\leq 2 \}$

Find all vectors $f_3(x) = a_0 + a_1 x + a_2 x^2$

such that

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x)$$

form a basis for $P_2$.

Soln: Need $f_1, f_2, f_3$ to span and be lin indep.

Equivalent to vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ below being a basis for $\mathbb{R}^3$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Basis $\iff a_2 \neq 0$

**Conclusion:** $f_3(x) = a_0 + a_1 x + a_2 x^2$ completes $f_1(x)$, $f_2(x)$ to a basis $\iff a_2 \neq 0$