Today: Office Hours 1-3 pm
3E2, E3
Lecture 8

Determinant Day

Welcome to

Friday: Quiz through 9:33
Review

of 2x2
determinants.

\[
A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}
\]

\[
\det(A) = \begin{vmatrix} a \\ c \end{vmatrix}
\]

Theorem (Geometric Interpretation of det)

\[
\det(A) = ad - bc
\]

\[
\text{Area of parallelogram with sides the row vectors of A}
\]

\[
\text{not put the row}
\]

\[
\text{det(A)}
\]
Let's see how both sides change under row ops.

Why is this true?

\[ \det(\mathbf{A}) \]

Area

\( \neq 0 \)

R3) scale by \( k \)

unchanged

unchanged

unchanged

unchanged

(\( R2 \)) exchange rows

to another row

(\( R1 \)) add

unchanged

unchanged

unchanged

unchanged

(\( \det \mathbf{A} \))

unchanged

unchanged

unchanged

unchanged

(\( R2 \)) exchange

unchanged

unchanged

unchanged

unchanged

R1) scale

unchanged

unchanged

unchanged

unchanged

(\( R2 \)) exchange

unchanged

unchanged

unchanged

unchanged

R3) scale by \( k \)

unchanged

unchanged

unchanged

unchanged
Assume $A \in \text{REF}$. To prove the theorem, it suffices now to prove

\[
\begin{array}{cccc}
| 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Possibilities

$$|\text{det}(A)|$$
2) Similar calculation for matrices in REF

Satisfying: 1) Similar behavior under row ops

We will introduce its determinant

Rest of lecture: for an n x n matrix,
Take $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Suppose we know det of matrices up to size $(n-1) \times (n-1)$.

Inductive Definition: $\det(A) = \prod_{i=1}^{n} a_{ii}$
\[
\det(A) = a_{11} \det(A_{11}) + a_{12} \det(A_{12}) + \ldots + a_{1n} \det(A_{1n}) + (-1)^{1+n} \det(A_{1n+1}) + \ldots \\
\]

Sum over row 1

Set \( A_{ij} \) = row matrix

\[
A_{ii} = (n-1) \times (n-1)
\]
\[
\begin{bmatrix}
-5 \\
1 \\
-2
\end{bmatrix}
\]

\[
= 1 \cdot (-6) - 2 \cdot (1) + 1 \cdot (3)
\]

\[
\det(A) = 1 \cdot \det\begin{bmatrix}
-2 \\
1
\end{bmatrix}
\]

\[
= \det\begin{bmatrix}
-1 & 0 & -2 \\
1 & 3 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

\[
= 8
\]

\[
\det(A) = 1 \cdot 2 - 2 \cdot (\cdot 3)
\]

Ex 1: Calc. \(\det(A)\) for following \(A\).
Key properties of determinant

1. Behavior under row ops

E: Element
A: A
A': \text{EA}
\[ \text{det}(A) = 1, 2, \ldots, n \]

\[ n \text{ pivots} \]

\[ A = \begin{bmatrix} 2 \text{ in REF} \\ \end{bmatrix} \]

\[ \text{det}(A) = 0 \]

\[ n \text{ pivots} \]

\[ \text{stand} \]

\[ A = \begin{bmatrix} 0 & \ldots & 0 & x \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & x \\ x & \ldots & x & x \end{bmatrix} \]
for lower Ω-ear matrices
We'll also see this is true
entries
then det(Λ) = product of diag entries.

Observe: If A is upper Ω-ear
Theorem (Geometric Interpretation) \[ \text{det}(\mathbf{A}) = \text{Volume of Parallelipiped} \]

With edges the rows of \( \mathbf{A} \).
For $A$ in $\text{REF}$, $\det(A) \neq 0$.

Thus it suffices to prove the theorem.

A invertible $\iff A^{-1}$ exists.

Proof: Apply row ops $A \rightarrow A$.

For $A$ invertible, $\det(A) \neq 0$.

Theorem 4
Theorem 5

In particular: $\det(A^n) = (\det(A))^n$

In particular: $\det(A) = \det(A^1) = \det(A)$.

If $A$ exists:

$\det(1^n) = 1$

Idea of proof: Expand $A'B$ as products of elem. matrices.

Then calculate by induction on # of elem. matrices of elem. matrices applying

Proof
Lin Indep
Span Rn & A
Cols of A
Lin Indep
Span Rn & B
Rows of A
Nice consequence
6. Theorem det(ATA) = det(A)
\[
A = \begin{vmatrix}
2 & 3 \\
4 & 5
\end{vmatrix}
\]

When \(n=2\), we can expand determinant to find: \(A = \begin{vmatrix}a & b \\
c & d\end{vmatrix} = ad - bc\)
\[ A = \text{Cofactor Expansion} \]

\[ \text{Cofactors } C_{ij} = (-1)^{i+j} \text{det}(A_{ij}) \]

For an \( n \times n \) matrix \( A \).
Theorem (Cofactor Expansion)

\[ \det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in} \]
(row i expansion)

\[ \det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj} \]
(col j expansion)

Note: row 1 expansion = definition of \( \det \)
\[
\det(A) = 1 \cdot (1) = 1.
\]

\[
\det(A) = (-1)(-3) = 3
\]

Exercise (Calculate \(\det(A)\) for)