<u>Welcome to Lecture 7</u> "How I learned to love <u>bases</u>"

<u>Friday</u>: Quiz through §2.6

<u>"All About That Bass"</u>

Warmup Exer 1) Which of the following lists of vectors is a basis for $\mathbb{R}^3$?

1) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}$ ✗ 2 vectors cannot span $\mathbb{R}^3$

2) $\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$ ✓

3) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ ✓

4) $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ ✗ 4 vectors cannot be lin indep in $\mathbb{R}^3$

<u>Exer 2)</u> Find bases for Null(A), Col(A) of $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

<u>Soln</u> Put A in REF

$$\rightsquigarrow \begin{array}{ccc} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{pivot cols} \quad \text{free col} \end{array}$$

<u>Null(A)</u> : Soln set $= \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \;\middle|\; x_3 \text{ any number} \right\}$

In param form $x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

<u>Basis</u>: $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Col(A) Take pivot cols from **original** matrix A

Basis $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underbrace{n = 3}_{\text{\# of cols}} = \underbrace{2}_{\substack{\text{\# of pivot cols} \\ \text{size of basis of Col(A)}}} + \underbrace{1}_{\substack{\text{\# of free cols} \\ \text{size of basis of Null(A)}}}$$

Exer 3) Find bases for Null(A), Col(A)

of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

pivot col   free col

Soln Aready in REF

Basis for Null(A):

$$\text{Soln set} = \left\{ \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} \right\}$$

x_2, x_3, x_4 any numbers

In param form

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Basis</u>: $\underline{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

<u>Bases for Col(A)</u> $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

col of original matrix A
corr. to pivot col of REF

$$n = 4 = \underbrace{3}_{\text{size of basis of Null(A)}} + \underbrace{1}_{\text{size of basis of Col(A)}}$$

<u>Def</u>: H a subspace of $\mathbb{R}^n$

we set $\dim H$ = size of any basis of H

<u>Fact</u>: size of basis is <u>independent</u> of particular basis

<u>Terminology</u>: A matrix

$\text{rank}(A) = \dim \text{Col}(A) =$ "# of pivots"

(<u>Very</u> <u>Important</u>) <u>Rank Theorem</u>:

A $m \times n$ matrix

Then $n = \operatorname{rank}(A) + \dim \operatorname{Null}(A)$

"# of cols = # of pivot cols + # of free cols"

<u>Note</u> 1) A surj $\iff \operatorname{rank}(A) = m$

2) A inj $\iff \operatorname{rank}(A) = n$

__Exer__ Find rank(A) for $A = \begin{bmatrix} c^2 & 2c-1 \\ 1 & 1 \end{bmatrix}$ as we vary c.

__Soln__ ... put in REF ... __Exer__: use this to solve

Let's check whether A is invertible

$\det(A) \neq 0 \iff A$ invertible

$$\det(A) = c^2 \cdot 1 - 1 \cdot (2c-1)$$
$$= c^2 - 2c + 1 = (c-1)^2$$

<u>If $c \neq 1$</u> then $\det(A) \neq 0$ so A invertible so $\mathrm{rank}(A) = 2$.

<u>If $c = 1$</u> then $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

so $\mathrm{rank}(A) = 1$.

(0 matrix is only matrix with rank = 0)

<u>What are bases</u> good <u>for</u>?

<u>Already seen</u>: use to define <u>dim</u>.

<u>N</u>ow we will use bases to define <u>coordinates</u> of vectors.

Let H be a subspace of $\mathbb{R}^n$

Let B be a <u>basis</u> $\underline{v}_1, \dots, \underline{v}_k$ of H

Let $\underline{v}$ be a vector in H

B <u>spans</u>: $\underline{v} = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k$

B <u>lin indep</u>: this implies that $a_1, \dots, a_k$ are uniquely determined

__Why__? Suppose $\underline{v} = a_1'\underline{v}_1 + \cdots + a_k'\underline{v}_k$ as well.

$$\underline{0} = \underline{v} - \underline{v} = (a_1 - a_1')\underline{v}_1 + \cdots + (a_k - a_k')\underline{v}_k$$

$$\xRightarrow[\underline{v}_1, \ldots, \underline{v}_k \text{ lin indep}]{} \quad a_1 - a_1' = \cdots = a_k - a_k' = 0$$

__Conclude__: $a_1 = a_1', \ldots$

$a_k = a_k'$

__Def__ The __coordinates__ of $\underline{v} \in H$ with respect to basis B of H $\quad B = \{\underline{v}_1, \dots, \underline{v}_k\}$

is the vector

$$[\underline{v}]_B = \left.\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}\right\} \dim H$$

where $\underline{v} = a_1\underline{v}_1 + \dots + a_k\underline{v}_k$

<u>Exer</u> Let $H = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{= \underline{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{= \underline{v}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{= \underline{v}_3} \right\}$

Observe $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are lin indep

Let B be basis $\underline{v}_1, \underline{v}_2, \underline{v}_3$ of H.

What are <u>coords</u> of $\underline{v} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ wrt B?

__Soln__ Solve lin syst

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 0 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Soln $\underline{x} = \begin{bmatrix} \cancel{3}\ 2 \\ 1 \\ 0 \end{bmatrix}$

Conclusion $\underline{v} = 2\underline{v}_1 + 1 \cdot \underline{v}_2 + 0 \cdot \underline{v}_3$

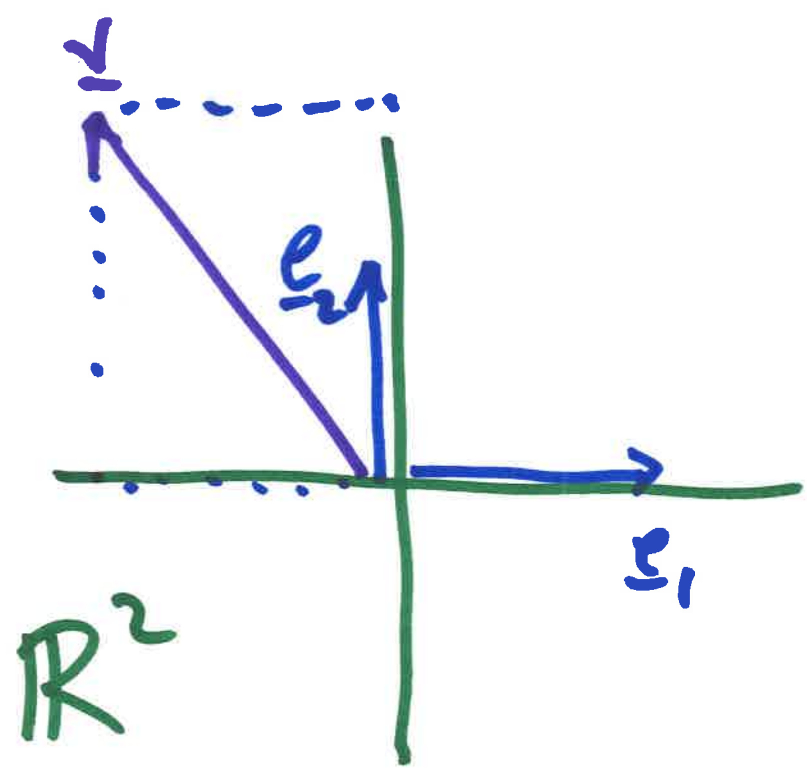
$$[\underline{v}]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

__Exer__ What are coords of any $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ wrt bases

1) $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

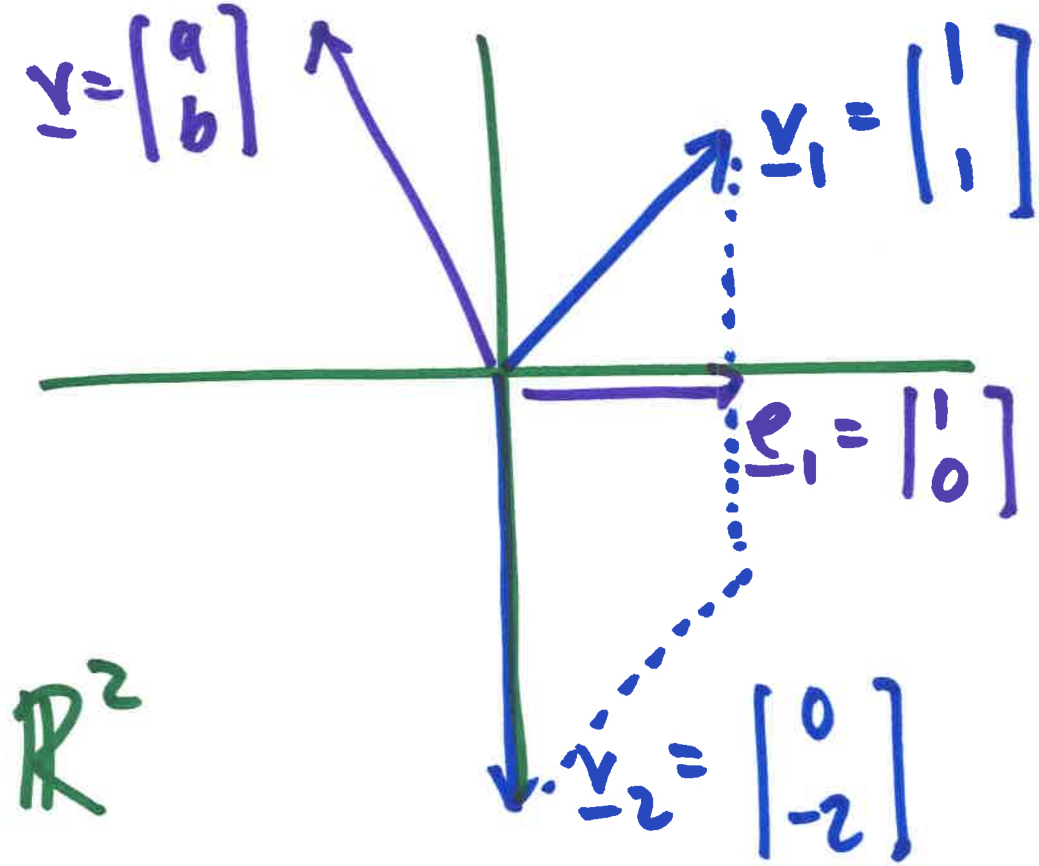
2) $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

__Soln__ 1)



$$[\underline{v}]_{\substack{\text{std}\\\text{basis}}} = \begin{bmatrix} a \\ b \end{bmatrix}$$

2)



Solve lin syst

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & -2 & b \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & -2 & b-a \end{array}\right]$$

Soln: $\underline{x} = \begin{bmatrix} a \\ \frac{b-a}{-2} \end{bmatrix}$

$$[\underline{v}]_B = \begin{bmatrix} a \\ \frac{b-a}{-2} \end{bmatrix}$$

Examples: i) $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow [\underline{v}]_B = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$

(ii) $\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow [\underline{v}]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$