Lecture 6 Inverses and Subspaces

Today: Office Hours 1-3pm 736 Evans

Friday Quiz through §2.6

Next week: Undergrad lectures at MSRI
by Edward Frenkel Mon 9/21
Tues 9/22

www.msri.org/workshops/804

__Warmup Exer__ 1) Find matrix of $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a-b \\ 3b \\ -a-2b \end{bmatrix}$

__Soln__

$$A = \underbrace{\left.\begin{bmatrix} | & | \\ T(\underline{e}_1) & T(\underline{e}_2) \\ | & | \end{bmatrix}\right\} 3}_{2} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & -2 \end{bmatrix}$$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T(\underline{e}_1) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad T(\underline{e}_2) = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

2) Find matrix of $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ (identity transf)

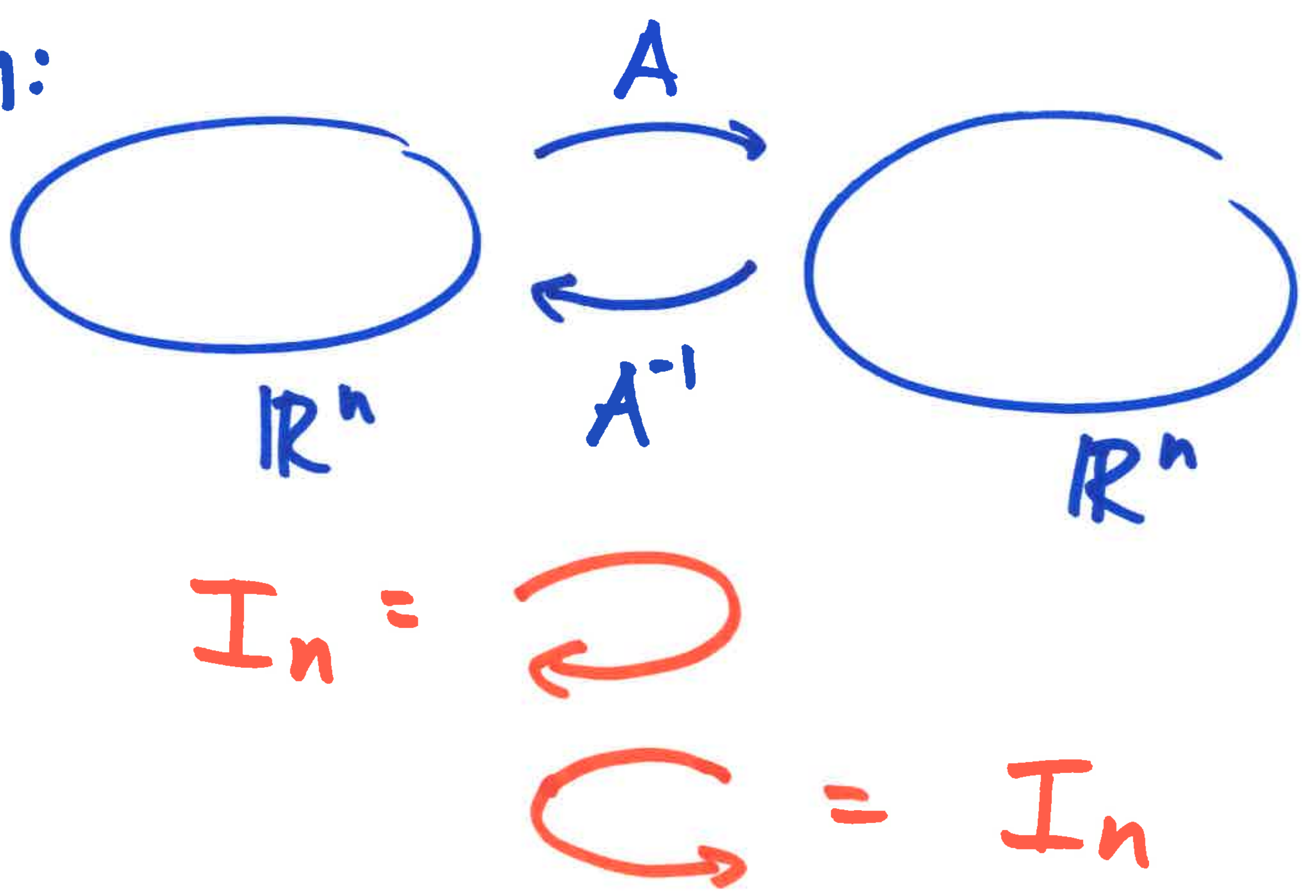
<u>Soln</u> $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Def</u>. $I_n = \left.\underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n}\right\} n$ $n \times n$ identity matrix

__Def__ A $n \times n$ matrix is __invertible__ if there is an $n \times n$ matrix A^{-1} such that

$$A A^{-1} = I_n = A^{-1} A$$

__Cartoon:__



Observations: 1) Def only makes sense for square matrices.

2) Inverse doesn't alway exist

If A is invertible, then lin trans defined by A must be inj and surj.

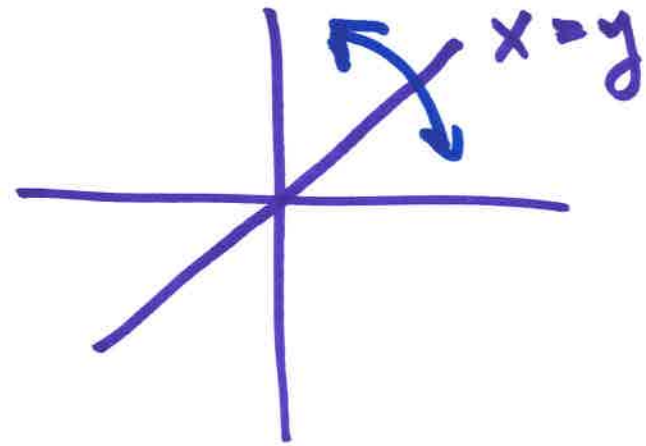
In fact: A invertible $\Longleftrightarrow$ A inj & surj.

Since A is square: A inj $\Longleftrightarrow$ A surj

Conclusion: A invertible $\Longleftrightarrow$ n pivots

__Exer__ Find inverse of A if it exists

1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2) $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ X not square!

3) $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

<u>Can check</u>: A is invertible since REF has 3 pivots

<u>Even better</u>: algorithm for inverse!

$$\left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{}_{A} \qquad \underbrace{}_{I_3}$$

Now reduce to RREF:

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

E_1A $\quad$ E_1

row swapped 1 & 2

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

E_2E_1A $\quad$ E_2E_1

subtracted row 1 from row 2

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$\underbrace{}_{E_3E_2E_1A} \qquad \underbrace{}_{E_3E_2E_1}$$

scaled row 3 by $\frac{1}{2}$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$I_3 = E_4E_3E_2E_1A$ $\qquad$ $A^{-1} = E_4E_3E_2E_1$

added row 3 to row 2

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ swaps rows 1 & 2

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ subtracts row 1 from row 2

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$ scales row 3 by $1/2$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ adds row 3 to row 2

__Remarks:__

$n \times n$ matrix

1) __Summary:__ A invertible $\iff$ REF has n pivots

If so, algorithm for inverse

2) __$n=2$__ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible $\iff$

determinant $\longrightarrow$ $\det(A) = ad-bc \neq 0$

If so, $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3) If inverse exists, it is unique.

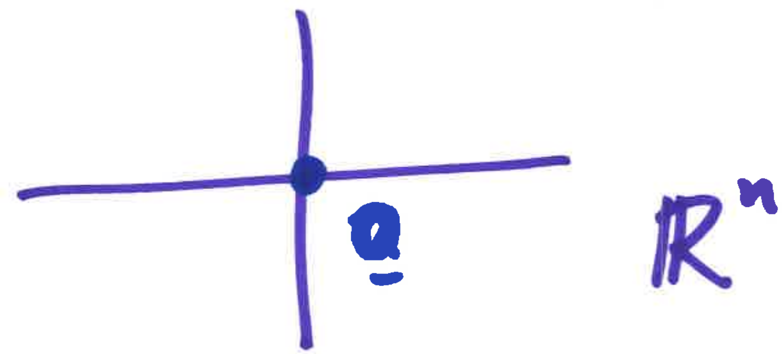
<u>Proof</u>. Suppose A_1^{-1}, A_2^{-1} are inverses of A.

$$A_2^{-1} = I A_2^{-1} = A_1^{-1} A A_2^{-1} = A_1^{-1} I = A_1^{-1}$$

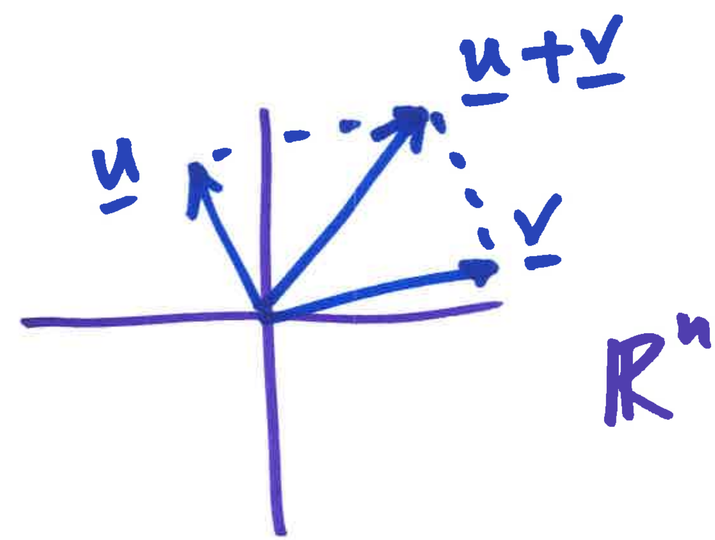
<u>Subspaces</u>

<u>Def</u> A <u>subspace</u> H of $\mathbb{R}^n$ is a subset of vectors of $\mathbb{R}^n$ satisfying:

1) $\underline{0} \in H$

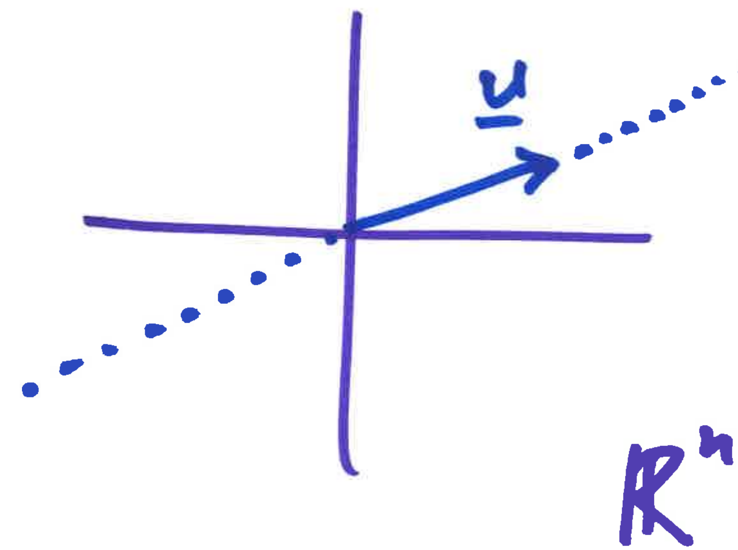


2) If $\underline{u}, \underline{v} \in H$ then $\underline{u} + \underline{v} \in H$



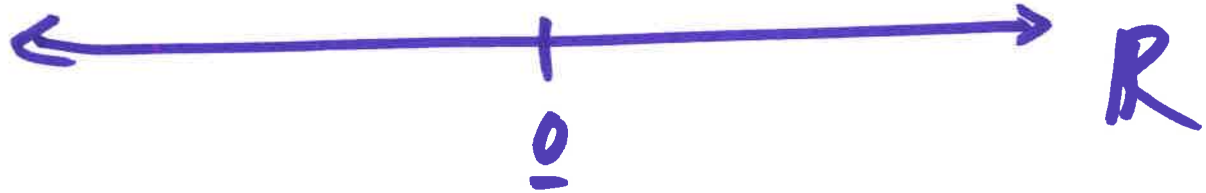
3) If $\underline{u} \in H$ and $c \in \mathbb{R}$

then $c \cdot \underline{u} \in H$



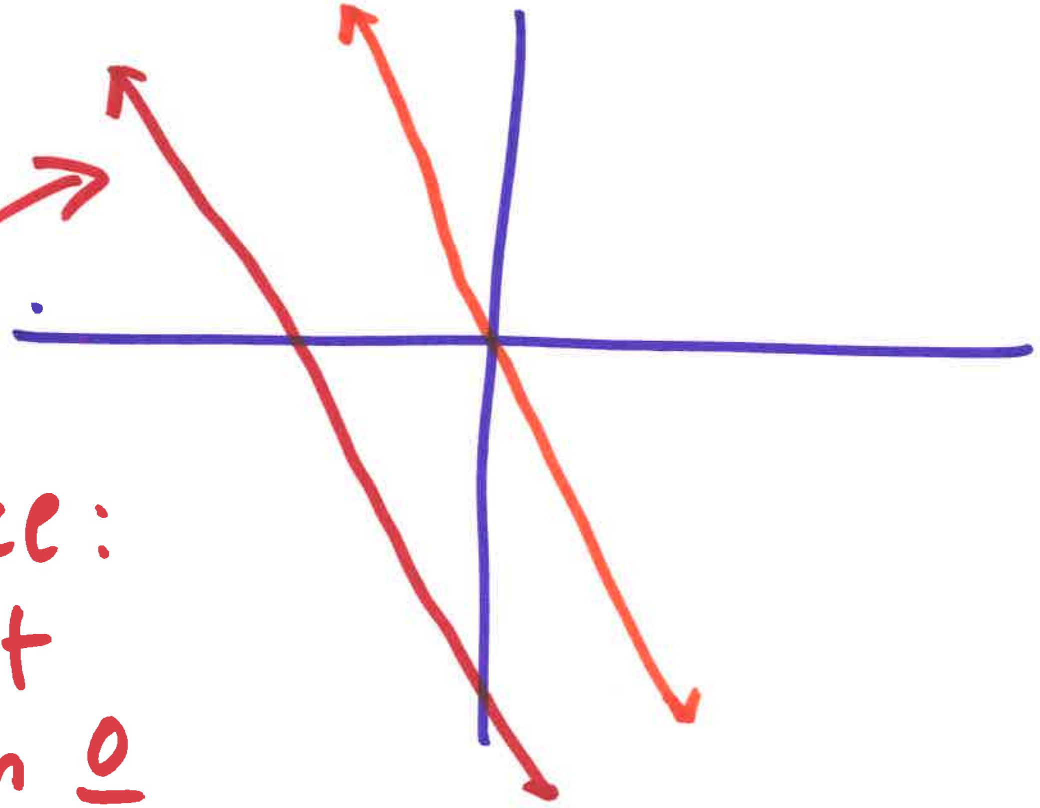
Examples in low dimensions:

<u>$n = 1$</u>



<u>Possibilities</u>: $\{\underline{0}\}$, $\mathbb{R}$

$\underline{n=2}$



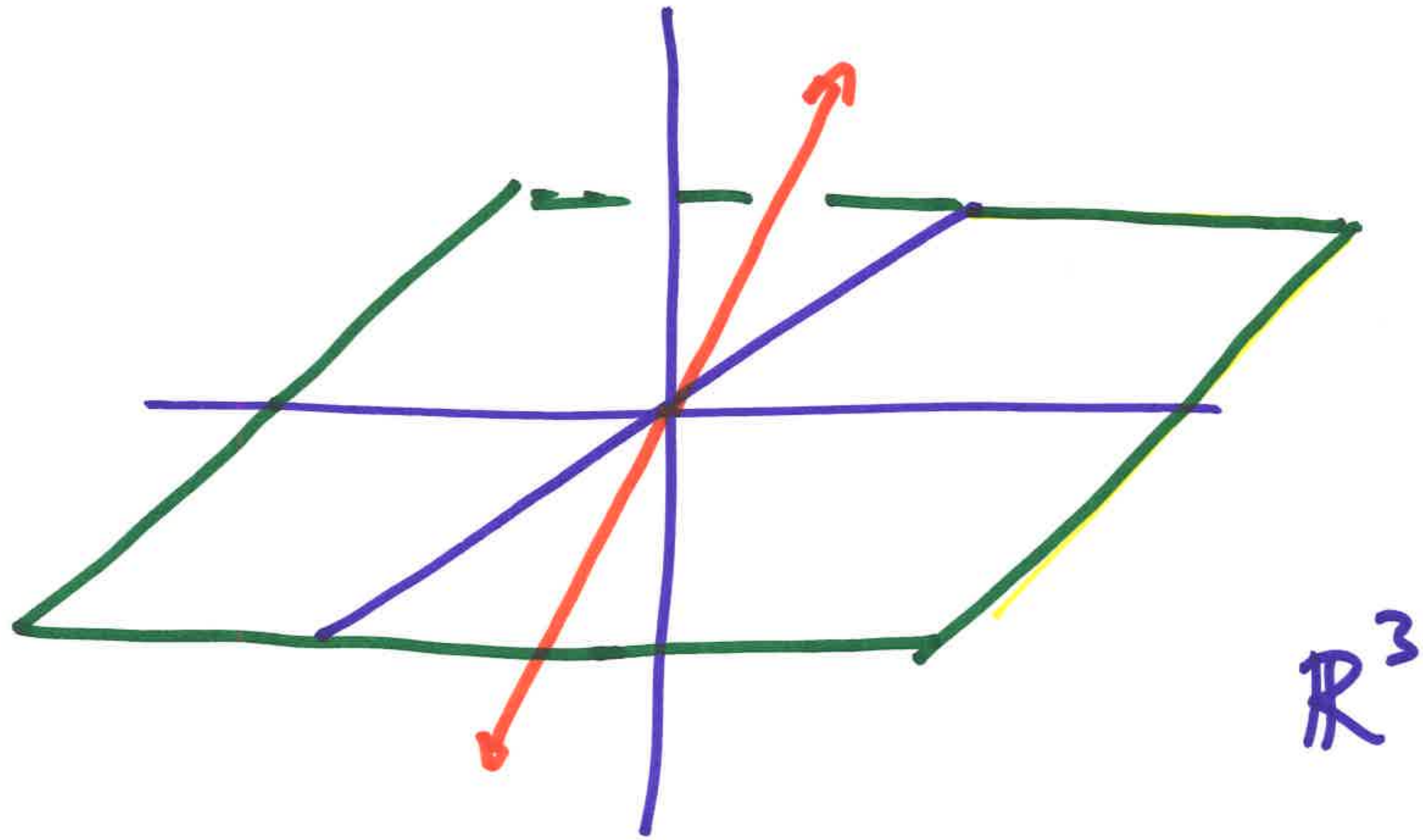
$\mathbb{R}^2$

not a <u>subspace</u>:
does not contain $\underline{0}$

<u>Possibilities</u>: $\{\underline{0}\}$, $\mathbb{R}^2$, any line through $\underline{0}$

also fails other requirements!

__$n=3$__



__Possibilities__: $\{\underline{0}\}$, $\mathbb{R}^3$, any line through $\underline{0}$, any plane through $\underline{0}$

Thm $\underline{v}_1, \ldots, \underline{v}_k$ be vectors in $\mathbb{R}^n$

Then $\operatorname{span}\{\underline{v}_1, \ldots, \underline{v}_k\}$ is a subspace of $\mathbb{R}^n$

In fact, it is the smallest subspace containing $\underline{v}_1, \ldots, \underline{v}_k$

Proof. Exercise.

Two most important examples:

A $m \times n$ matrix

inside $\mathbb{R}^m$

Def 1) Column space $\mathrm{Col}(A)$ = span of cols of A

(= image of A)

2) Null space $\mathrm{Null}(A)$ = soln set of $A\underline{x} = \underline{0}$

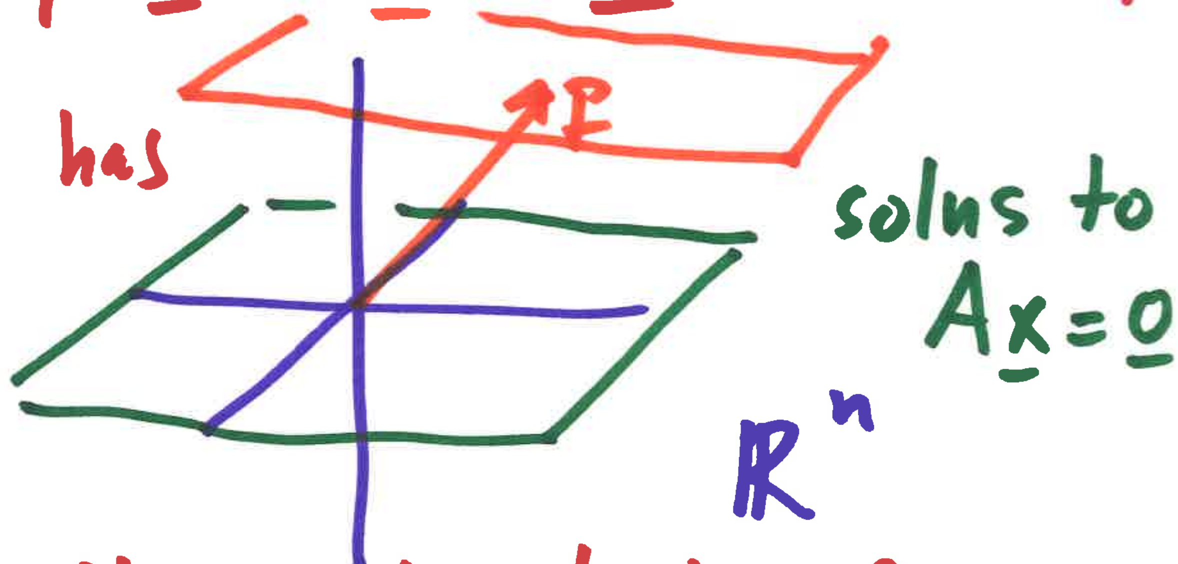
inside $\mathbb{R}^n$

Theorem $\mathrm{Col}(A)$, $\mathrm{Null}(A)$ are subspaces

Proof. Exercise.

Caution: soln set to $A\underline{x} = \underline{b}$ when $\underline{b} \neq \underline{0}$ is not a subspace.

1) Either $A\underline{x} = \underline{b}$ has no solns



2) Solns are $\underline{p} + H$ where H are solns to $A\underline{x} = \underline{0}$

(Very important) **Def** A basis for a subspace H of $\mathbb{R}^n$ is an ordered list of vectors $\underline{v}_1, \ldots, \underline{v}_k$ so that

$\boxed{\text{Goldilocks \& 3 Bears}}$

1) $\underline{v}_1, \ldots, \underline{v}_k$ spans H

(big enough) if $\underline{v} \in H$, then $\underline{v} = a_1\underline{v}_1 + \cdots + a_k\underline{v}_k$

2) $\underline{v}_1, \ldots, \underline{v}_k$ are lin indep

(not too big) if $\underline{0} = a_1\underline{v}_1 + \cdots + a_k\underline{v}_k$, then $a_1 = \cdots = a_k = 0$