Especially if it's an answer in the back of a textbook...

Question Authority!

Friday Quiz through §1.7

Welcome to Lecture 5! Intro to Matrix Algebra
3) matrix even \( A \bar{x} = \bar{b} \)

\[
\begin{pmatrix}
\underbrace{A}_{m \times n} \\
\underbrace{\bar{b}}_{n 	imes 1}
\end{pmatrix}
\]

2) augmented matrix

\[ \text{in n vars} \]

1) linear system of m e.qns

Three equivalent concepts:

Before things get more abstract, let's review where we've been.
The following are equivalent:

1) There is soln to \( Ax = b \).
2) REF of \([A : b]\) has no pivot in augmentation col.
3) \( b \in \text{Image}\ (A) = \text{Span}\ \{\text{cols of } A\}\)

\[ R^n \xrightarrow{A} R^m \]
1) Existence

(+) 

3) \( R^n = \text{Image}(A) = \text{Span}\text{ columns of } A^T \)

Every row of \( A \) has a pivot in REF.

2) \( \text{REF of } A \) has pivot in every row.

1) There is a solution to \( Ax = b \) for any \( b \)

The following are equivalent:
4) (cols of A are lin indep.

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Triv. soln x = 0

3) A x = 0 has only the

Triv. soln x = 0

2) REF of A has a pivot

in every col. (no free vars.)

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If A x = b has a soln,

Then soln is unique

Then soln is unique

The following are equivalent:

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Thus $i)$ holds: $x = y$.

By inequality $Ax - Ay = 4(x - y)$

$\bar{x} - \bar{y} = 0$ since $A(x - y) = 0$

By definition: If $3)$ holds, then

Conclusion: If $3)$ holds, then

By inequality $Ax - Ay = 4(x - y)$

Suppose $Ax = \bar{b} = Ay$ so $Ax - Ay = 0$

Key idea: map given by $A$ is linear.

Why are (1) $\iff$ (3) equivalent?
Always represented by unique matrix:

\[ A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \]

\[ e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \]

Continue with \( \text{lin transf} \ T: \mathbb{R}^n \rightarrow \mathbb{R}^n \)
Then $A\bar{y} = i_{th}$ col of $A$

and $\bar{v}$ is a vector $\bar{v} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

Key idea: If $A$ is a mxn matrix:

Why is this true?
Exer 1) Find the matrix for linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by 90° clockwise rotation.
2) Find matrix for lin transf onto 2nd and 3rd coordinates.

\[ T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \text{ given by projection} \]

\[ T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

\[ T \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \]
\[
A = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} \\
0 & \frac{1}{2} & \frac{1}{3} \\
0 & 0 & 1
\end{bmatrix}
\]

Matrix of T is
we have \( x = \frac{y}{y} \).

\[ T \bar{x} = \frac{y}{y} \text{ for } x, y \in \mathbb{R}^n \]

\( T \bar{x} = \frac{y}{y} \) whenever

\[ \bar{b} = T \bar{x} \]

is image of some \( x \in \mathbb{R}^n \)

is onto and surjective if every \( \bar{b} \in \mathbb{R}^m \)

(1) onto / surjective if every \( \bar{b} \in \mathbb{R}^m \)

(2) one-to-one / injective if whenever

\[ x = y \implies \bar{x} = \bar{y} \]
2) one-to-one / injective

onto / surjective

Cartoon
Conclusion: not sure!

A pivot in each row.

Since $2 \geq 3$, there cannot be.

For sure? What is $\text{Image}(A)$?

Explain:

$$A = \begin{bmatrix} 1 & -1 \ 1 & -2 \ 1 & 2 \end{bmatrix}$$

Given by
Conclusion: In the pivot in each column, REF check there is a pivot. What are solutions to \( \overline{A}x = 0 \)? For instance:
A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is surjective if and only if its matrix $A$ has a right inverse.

Furthermore, if $A$ is a square matrix, then $A$ is invertible if and only if its matrix $A$ has a right inverse.
in each col \( T \) is ins. \( \iff \) \( \text{P.E.F has pivot} \iff m=n \)

in every row

\( \text{P.E.F has pivot} \iff \text{is surj.} \iff \text{T is ins.} \)

Then \( \text{T is surj.} \iff \text{n is transf.} \)

\( \text{Theorem: T:} \ \mathbb{R}^n \rightarrow \mathbb{R}^n \)

Special case when \( m=n \)
Matrix Algebra

What can we do with matrices?

1) **add**: $A, B$ $m \times n$ matrices

   $A + B$ $m \times n$ matrix with entries

   $$(A + B)_{ij} = A_{ij} + B_{ij}$$

2) **scale**: $A$ $m \times n$ matrix, $c$ number

   $cA$ $m \times n$ matrix with entries

   $$(cA)_{ij} = cA_{ij}$$
where \( B = \begin{bmatrix} \frac{1}{p} & \cdots & \frac{1}{p} \\ \end{bmatrix} \)

\[
\begin{pmatrix}
\Phi \\
p \\
\vdots \\
m
\end{pmatrix}
\]

\[
AB = \begin{bmatrix} A_{b1} & \cdots & A_{bn} \\ \end{bmatrix}
\]

AB \text{ mxp matrix given by}

3) multiply A mxn, B nxp
AB is composition of first B followed by A
Exer: Calculate $AB$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$.
Matrix algebra has many good properties

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \]

1. \( AB \neq BA \)

Caution:

... See book!
\[
\begin{bmatrix}
  0 & -3 \\
  0 & 0
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  2
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}
\]

A = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
B = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
C = 2

3) A \neq B = A \neq C = B = C

\[
\begin{bmatrix}
  0 & 3 \\
  0 & 0
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  2
\end{bmatrix} = \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix}
\]

A = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
B = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
B = 2

(2) AB = 0 \Rightarrow A = 0 \text{ or } B = 0