<u>Lecture</u> 4 Reinterpreting lin systs. in terms of <u>lin. transformations</u>

<u>Today</u> Office Hours 1-3pm
736 Evans

<u>Friday</u> Quiz through §1.7

Hang in there!

<u>Warmup</u> Which subsets of the following list are lin indep:

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \underline{v}_5 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Complete soln will be left to you but we'll get started together.

1 element subsets

A single vector $\underline{v}$ is lin indep if

$$a \cdot \underline{v} = 0 \implies a = 0$$

for any a.

Thus $\underline{v}$ is lin indep if and only if $\underline{v} \neq \underline{0}$.

Any single vector $\underline{v}_1$, $\underline{v}_3$, $\underline{v}_4$, or $\underline{v}_5$ is lin indep but $\underline{v}_2$ is not

<u>2 element subsets</u>

Two vectors $\underline{u}_1, \underline{u}_2$ are <u>lin indep</u> if $a_1\underline{u}_1 + a_2\underline{u}_2 = 0$ implies $a_1 = a_2 = 0$

<u>Geometrically</u>: two vectors are lin indep if and only if they are <u>not</u> colinear

In our exercise, let's try

$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Take $a_1 = 0, a_2 = 1$

so $a_1\underline{v}_1 + a_2\underline{v}_2 = \underline{0}$

Lin depend!

$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$a_1\underline{v}_1 + a_3\underline{v}_3 =$

$\begin{bmatrix} a_1 + a_3 \\ -a_3 \\ -a_1 \end{bmatrix} = \underline{0}$

Must have $a_1 = a_3 = 0$

Lin Indep!

Let's do example of 3 elt subset

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \underline{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Suppose $a_1 \underline{v}_1 + a_3 \underline{v}_3 + a_4 \underline{v}_4 = \underline{0}$

So $$\begin{bmatrix} a_1 + a_3 + a_4 \\ -a_3 + a_4 \\ -a_1 + a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Check</u>: implies $a_1 = a_3 = a_4 = 0$

Lin Indep!

Span	Lin Indep
Want many vectors in small space	Want few vectors in big space
$\underline{v}_1, \dots, \underline{v}_k$ span all of $\mathbb{R}^m$ $\Longrightarrow$ $k \geq m$	$\underline{v}_1, \dots, \underline{v}_k$ lin indep in $\mathbb{R}^m$ $\Longrightarrow$ $k \leq m$
Adding vectors to list only helps	Deleting vectors from list only helps
$\underline{v}_1, \dots, \underline{v}_k$ span all of $\mathbb{R}^m$ $\Longleftrightarrow$ lin syst $\left[\begin{array}{ccc:c} \vert & & \vert & \\ \underline{v}_1 & \cdots & \underline{v}_k & \underline{b} \\ \vert & & \vert & \end{array}\right]$ has soln for all $\underline{b}$	$\underline{v}_1, \dots, \underline{v}_k$ lin indep in $\mathbb{R}^m$ $\Longleftrightarrow$ lin syst $\left[\begin{array}{ccc:c} \vert & & \vert & \\ \underline{v}_1 & \cdots & \underline{v}_k & \underline{0} \\ \vert & & \vert & \end{array}\right]$ has unique soln $\underline{0}$

Theorem $\underline{v}_1, \dots, \underline{v}_k$ is lin dependent

$$\iff$$

there is an index i so that

$$\underline{v}_i \in \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_{i-1}\}$$

Remark: If lin dependent then there always is smallest such index i

Exer Which of the following vectors is first to be in span of preceding vectors?

$$\underline{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \quad \underline{v}_4 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix} \quad \underline{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

Soln Want $a_1 \underline{v}_1 + \cdots + a_i \underline{v}_i = \underline{0}$

with $a_i \neq 0$ (for example $a_i = 1$)

And no such lin depend with smaller i.

Put vectors in matrix

$$\begin{bmatrix} 2 & -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 2 & -2 \end{bmatrix} \xrightarrow{\text{row ops}} \underbrace{\begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}}_{\text{REF form}}$$

No pivot in 3rd col!

Observe that a_3 is <u>first free var.</u>

Set $a_3 = 1$ (or any nonzero number)

Set $a_4 = a_5 = 0$

Solve for $a_1 = -1$
$a_2 = -2$

<u>Conclusion</u>: first lin depend $-1 \cdot \underline{v}_1 + (-2) \cdot \underline{v}_2 + \underline{v}_3 = \underline{0}$

By Theorem, $\underline{v}_3$ is first vector in span of preceding vectors

New terminology: linear transformations

Product of a matrix and vector

Input: A $m \times n$ matrix $\qquad A = \begin{bmatrix} | & & | \\ \underline{a}_1 & \cdots & \underline{a}_n \\ | & & | \end{bmatrix}$

$\underline{x}$ n-vector $\qquad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Output $\underline{b}$ m-vector $\qquad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$$\underline{b} = \begin{bmatrix} | & & | \\ \underline{a}_1 & \cdots & \underline{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$$

Example $A = \begin{bmatrix} 2 & 0 & 3 \\ 7 & -1 & 5 \end{bmatrix}$ $\underline{x} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$$\underline{b} = A\underline{x} = \begin{bmatrix} 2 & 0 & 3 \\ 7 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 1 + 0\cdot 1 + 3\cdot(-3) \\ 7\cdot 1 + (-1)\cdot 1 + 5(-3) \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$$

Think about matrix product as
<u>map</u> / <u>mapping</u> / <u>transformation</u>

$$\underset{\text{domain}}{\mathbb{R}^n} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^m}$$

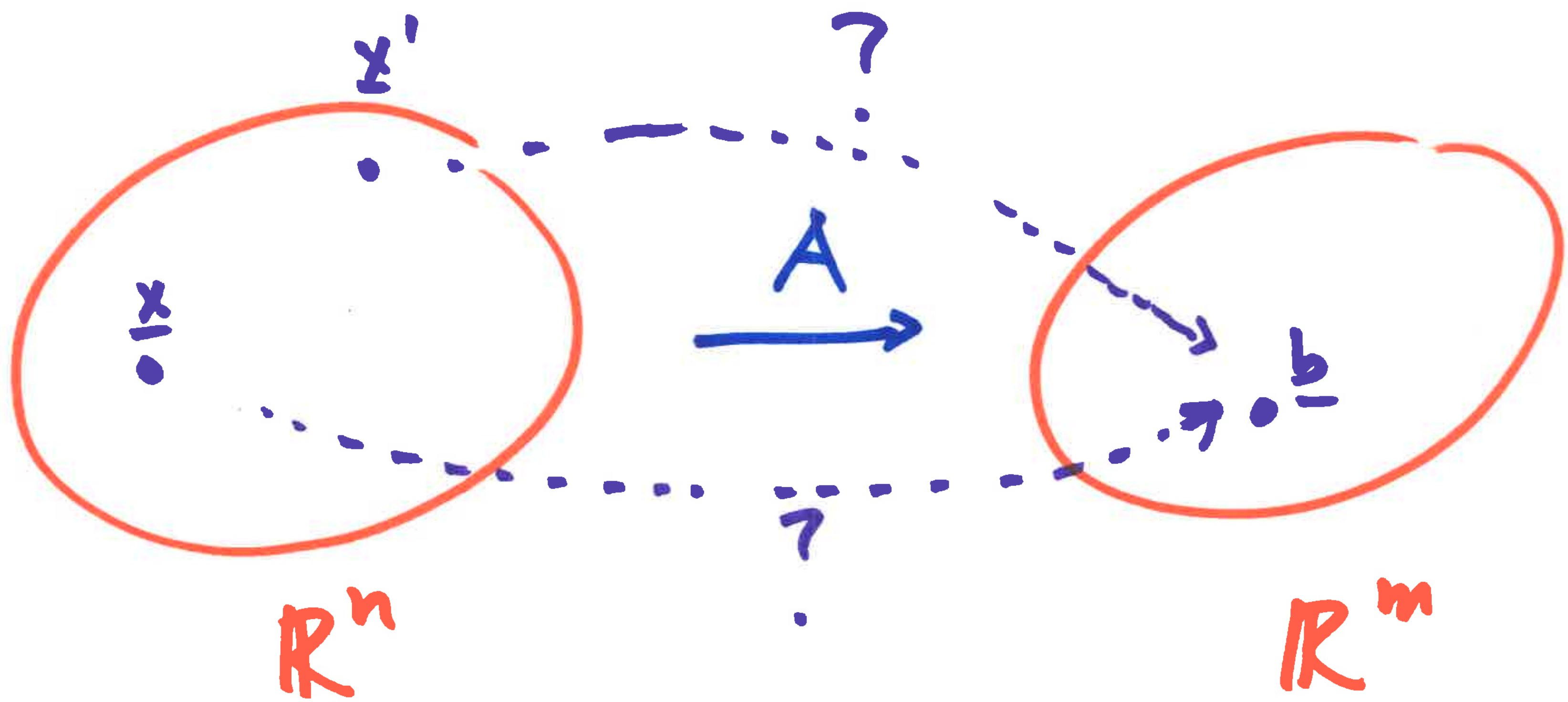
$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \underline{b} = A\underline{x} = x_1\underline{a}_1 + \cdots + x_n\underline{a}_n$$

Solving eqn $A\underline{x} = \underline{b}$ becomes answering questions:

1) <u>Existence of soln</u>: is there $\underline{x} \in \mathbb{R}^n$ that A takes to $\underline{b} \in \mathbb{R}^m$?

2) <u>Uniqueness of soln</u>: how many $\underline{x} \in \mathbb{R}^n$ does A take to $\underline{b} \in \mathbb{R}^m$?

Cartoon

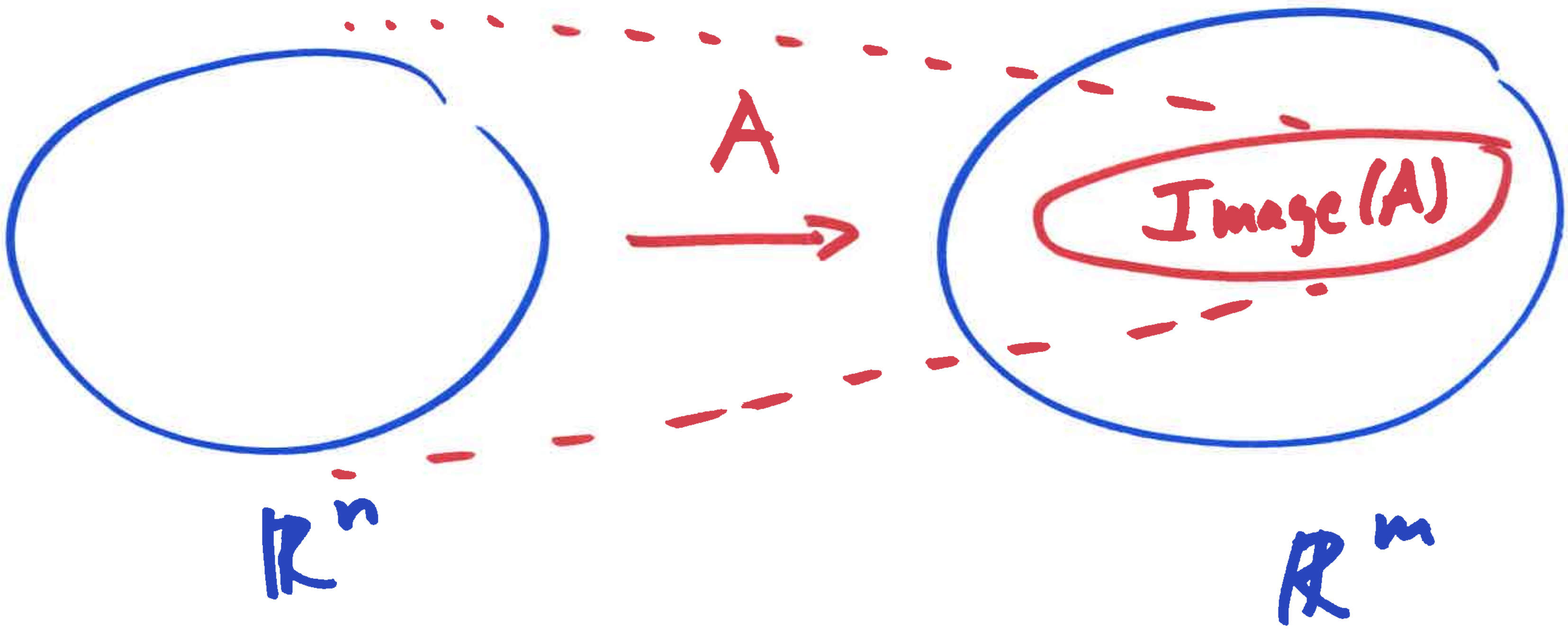


Def The _image_ / _range_ of the map given by A is

$$\text{Image}(A) = \{ \underline{b} \in \mathbb{R}^m \text{ so that there is } \underline{x} \in \mathbb{R}^n \text{ with } A\underline{x} = \underline{b} \}$$

Observe $\text{Image}(A)$ = span of cols of A

Cartoon of Image



Exer Is $\underline{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in image of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} ?$$

Soln Solve $\left[\begin{array}{ccc:c} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right]$

... check $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ solves.

What is special about map given by a matrix A?

1) $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$

2) $A(c\cdot\underline{x}) = c\cdot(A\underline{x})$

<u>Def</u> A map $T:\mathbb{R}^n \to \mathbb{R}^m$ is called a <u>linear map</u> / linear transformation if it satisfies properties 1) and 2)

<u>Amazing fact</u>: Any lin. transf. $T: \mathbb{R}^n \to \mathbb{R}^m$ is given by a unique matrix

$$A = \begin{bmatrix} | & & | \\ T(\underline{e}_1) & \cdots & T(\underline{e}_n) \\ | & & | \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$