Please remember course evaluation.

Exam: Wed 12/16 3-6pm Wheeler

Office Hours: TBA

Next week: Reviews during usual lecture

Friday Quiz through 9/10.3

Fourier Series!

Sorrow... but we'll always have Lecture 26: Partition is such sweet
Exer Calculate Fourier sine series

for $f(x) = 1$ on $[0, \pi]$

$\sum_{n=1}^{\infty} c_n \sin(nx)$

Calculate!
\[ \int_0^{\frac{\pi}{2}} \sin(mx) \sin(nx) \, dx = 0 \quad \text{if } m \neq n \]

\[ \langle \sin(mx), \sin(nx) \rangle = \begin{cases} \frac{\pi}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

Orthogonal set of functions on \([0, \pi]\):

\[ \{ \sin(nx) : n = 1, 2, 3, \ldots \} \]
Formula for coeffs:

\[ C_n = \frac{\langle f, \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle} \]

\[ = \frac{1}{\frac{\pi}{2}} \int_0^\pi f(x) \cdot \sin(nx) \, dx \]

\[ = \frac{2}{\pi} \int_0^\pi 1 \cdot \sin(nx) \, dx \]
\[
\begin{align*}
\frac{1}{2} & = \frac{1}{2} \\
\frac{1}{2} & = \frac{1}{2} \\
\frac{1}{2} & = \frac{1}{2} \\
(1 - \cos(nx)) & = \frac{1}{2} \\
\end{align*}
\]

\[
\begin{align*}
\text{y} & = \text{even} \\
\text{z} & = \text{odd} \\
\text{x} & = \text{even} \\
\end{align*}
\]
\( \ldots + \frac{5}{\sin(5x)} + \frac{3}{\sin(3x)} + \sin(x) \) \( \frac{\pi}{4} = \frac{FS}{\text{conclusion}} \)
So
\[ |T| = \frac{1}{2} \]

\[ F_2 x = \frac{1}{2} \]

\[ x = \frac{1}{2} \]

\[ \eta = \frac{1}{3} \]

\[ f(x) = \frac{1}{2} \]

\[ f(\frac{1}{2}) = \frac{1}{2} \]

\[ \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \]

\[ \text{Wow!} \]

Let's enjoy this for a moment.
What the heck is going on?

$$\begin{align*}
0 &= \frac{1}{h} (0 + 0 + 0 + 0 + 0) \\
\text{So } f''(0) &= 0 \\
f(0) &= 1
\end{align*}$$

But what if we evaluate at $x=0$?
So what can we do?

but if it is a nice fn

... \[ f(0) = 1 \] ...

In exercise, \( f(0) = 1 \) ...

then \( f = \text{Fourier sine series} \)

and \( f \) pw cont

Recall Thm \( f: [a, b] \to \mathbb{R} \) wth
\[ f(x) = f(x + 7) \]

**Def:** \( f : \mathbb{R} \rightarrow \mathbb{R} \) is \( \text{periodic} \). Periodic functions on \( \mathbb{R} \).

Fourier series not really about functions on \([0, L]\) but are about periodic functions on \( \mathbb{R} \).
2L

Periodic of length

Ex: \sqrt{\frac{7}{3\pi}} \sin \left( \frac{\pi x}{2L} \right)
Periodic of length $\frac{2L}{n}$

In particular

$$\left(\frac{7}{x\pi n}\right)\cos\left(\frac{7}{x\pi n}\right)$$

$$\left(\frac{7}{x\pi n}\right)\sin\left(\frac{7}{x\pi n}\right)$$

$2)$ $(\cos\left(\frac{7}{x\pi n}\right))$
on \([0, L]\) should also extend to 2\(\pi\)-periodic Fourier series. Suppose

\[
\sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{L} x\right) = f(x) = \lim_{n \to \infty} \left[ \sum_{n=1}^{n} c_n \sin\left(\frac{\pi n}{L} x\right) \right]
\]

Return to Fourier sine series:
In previous exam: \( f(x) = 1 \) \( \forall x \in \mathbb{R} \) for every even number.

But still have problem:

\[
\sin \left( \frac{\pi x}{2} \right) = 0 \quad f(0) = 1 \neq \frac{\pi}{2}
\]

Of course: \( f(0) = 1 \).
\[
\frac{7}{\pi x} \sin \left( \frac{\pi n}{x} \right) \quad \text{for even } \frac{\pi n}{x} \cos \left( \frac{\pi n}{x} \right) \quad \text{for odd } \frac{\pi n}{x}
\]

\[
\text{Ex:} \quad \begin{cases} 
\int_{1}^{2} f(x) \, dx = -f(x) \\
\int_{0}^{\pi} f(-x) = f(x) \\
\text{Def: } f(x) \text{ is } \mathbb{R} \rightarrow \mathbb{R}
\end{cases}
\]
To extend \( f(x) = 1 \) on \( \mathbb{Z} \), we need to take the periodic and odd extension. We cannot take the periodic and odd extension.
Suggestion: If we can write odd

$$f''(x) + f(x) = f(x) \text{ even } f(x)$$

a sum of an even and odd

Exercise: Show any $f: \mathbb{R} \rightarrow \mathbb{R}$ is uniquely
we can handle all far.
\[ \langle f, g \rangle = \int_{-L}^{L} f(x) \overline{g(x)} \, dx \]

with respect to inner product

\[ \sin \left( \frac{\pi n x}{L} \right) \quad n = 1, 2, 3, \ldots \]
\[ \cos \left( \frac{\pi n x}{L} \right) \quad n = 0, 1, 2, \ldots \]

For thus on \([-L, L]\), we have...
\[ 0 = u = m \quad 7 \quad 2 \]

\[ 0 < \quad n = m \quad 7 \quad 0 \quad \left\{ \begin{array}{ll}
(\frac{7}{x \pi n}) \cos (\frac{7}{x \pi m}) & > 0 \\
n \neq m & \quad 7 \\
n \neq m & \quad 0 \quad \left\{ \begin{array}{ll}
(\frac{7}{x \pi n}) \sin (\frac{7}{x \pi m}) & > 0 \\
n \neq m & \quad 0 \\
n \neq m & \quad 11 \quad m' \quad 0 = \quad \left\{ \begin{array}{ll}
(\frac{7}{x \pi n}) \sin (\frac{7}{x \pi m}) & \cos (\frac{7}{x \pi m})
\end{array} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right.
Define Fourier coefficients of \( f \) on \([-\pi, \pi] \):

\[
\begin{align*}
\hat{a}_n & = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \\
\hat{b}_n & = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx
\end{align*}
\]
\[
\frac{x}{\pi n} \left( \frac{7}{x \pi n} \right) \sin (x f) f \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{7}{1} \right) = b_n
\]

\[
0 = a_n \left[ x \left( \frac{7}{x \pi n} \right) \cos (x f) f \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{7}{1} \right) \right]
\]

More explicitly:

\[
\forall n = 1, 2, 3, \ldots
\]
Caution: Some people such as the book's authors, define $a_0$ to be twice our $a_0$ and then divide by two...

$$\sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n \pi x}{L} \right) + b_n \sin \left( \frac{n \pi x}{L} \right) \right) + a_0$$

Defining Fourier cos and sin services
Then $f : \mathbb{R} \to \mathbb{R}$ periodic of length $2L$

Fourier series and sine and cosine:

$$f(x) = \frac{1}{2} \left( f(0) + \sum_{n \neq 0} \left( f(x) - f(x + 2nL) \right) \right)$$

$x + a \leftrightarrow f \text{ cont.}$

in general