Lecture 26 Parting is such sweet sorrow ... but we'll always have Fourier Series!

Friday Quiz through §10.3

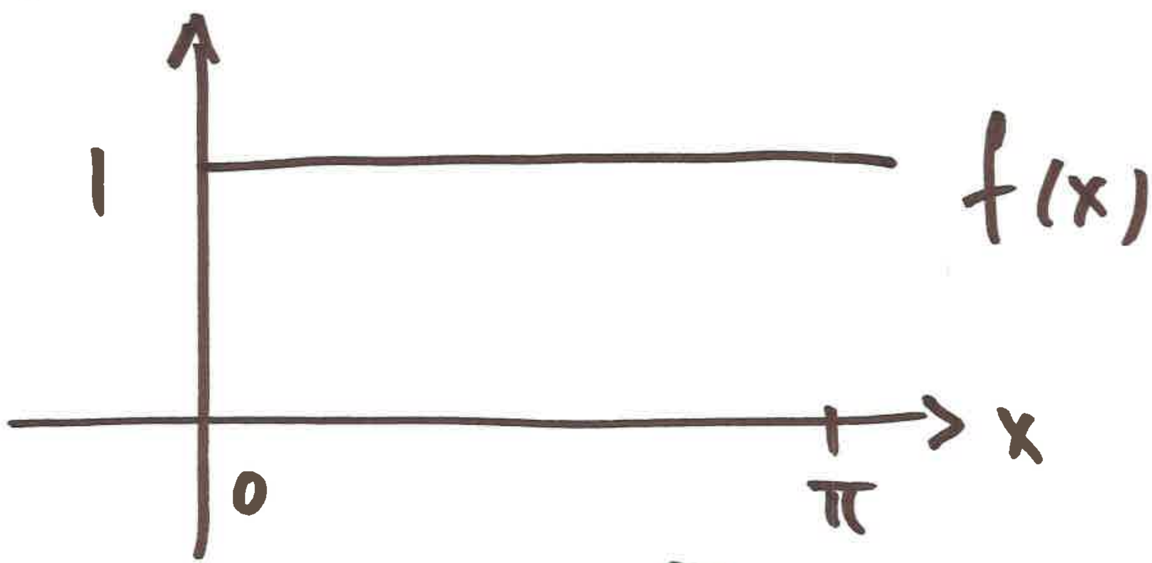
Next week Reviews during usual lecture meetings

Office Hours : TBA

Exam : Wed 12/16 3-6pm Wheeler

Please remember course evaluation!

__Exer__ Calculate Fourier sine series for $f(x) = 1$ on $[0, \pi]$



__Soln__ $\text{F.S} = \sum_{n=1}^{\infty} c_n \sin(nx)$

calculate!

Recall: $\sin(nx)$ $n = 1, 2, 3, \ldots$

are orthogonal set of fns on $[0, \pi]$

$$\langle \sin(mx), \sin(nx) \rangle = \int_0^\pi \sin(mx)\sin(nx)\,dx$$

$$= \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \end{cases}$$

orthogonal

Formula for coeffs:

$$c_n = \frac{\langle f, \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle}$$

$$= \frac{\int_0^{\pi} f(x) \cdot \sin(nx)\, dx}{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \begin{cases} -\frac{1}{n} - \left(-\frac{1}{n}\right) & n \text{ even} \\ \frac{1}{n} - \left(-\frac{1}{n}\right) & n \text{ odd} \end{cases}$$

$$= \frac{2}{\pi} \begin{cases} 0 & n \text{ even} \\ \frac{2}{n} & n \text{ odd} \end{cases}$$

Conclusion

$$FS = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right)$$

Let's enjoy this for a moment:

$$f\left(\frac{\pi}{2}\right) = 1$$

FS at $x = \frac{\pi}{2}$ $= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$

So $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ Wow!

But what if we evaluate at $\underline{x=0}$?

$$f(0) = 1$$

$$\text{FS at } x=0 = \frac{4}{\pi}\left(0 + 0 + 0 + \dots\right)$$

$$= 0$$

What the heck is going on?

Recall Thm $f : [0, L] \to \mathbb{R}$ with

$f(0) = 0 = f(L)$ and f' pw cont

then f = Fourier sine series

In exercise, $f(0) = 1$ . . .

but it is a nice fn

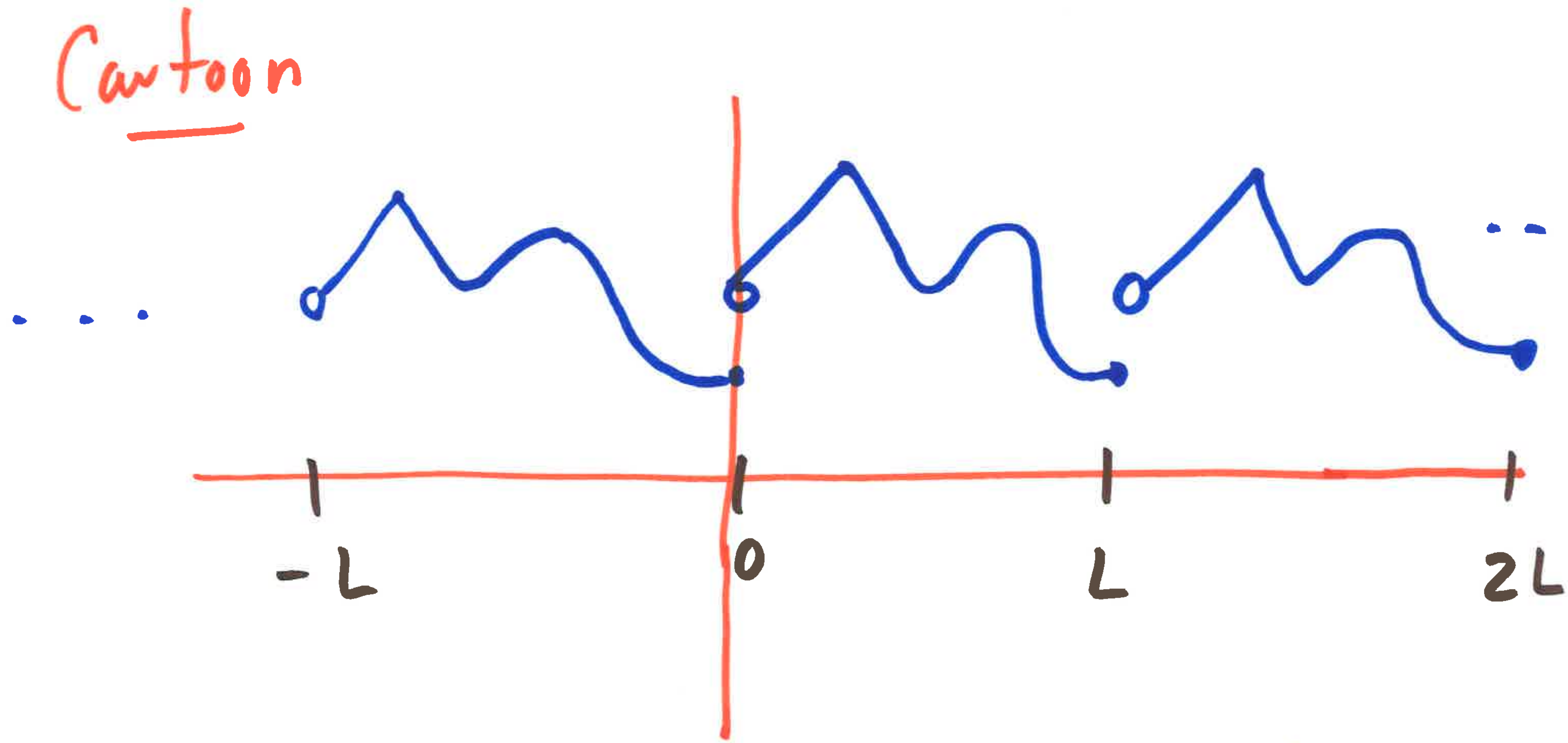
so what can we do?

Fourier series are not really about fns on $[0,L]$ but are about periodic fns on $\mathbb{R}$.

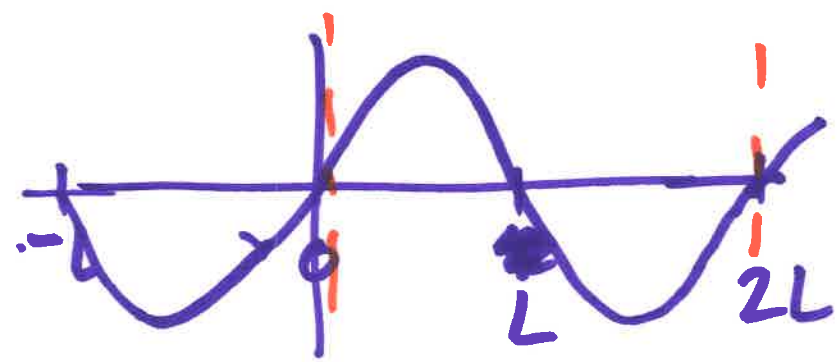
Def: $f:\mathbb{R}\to\mathbb{R}$ is periodic of length L if

$$f(x+L) = f(x)$$

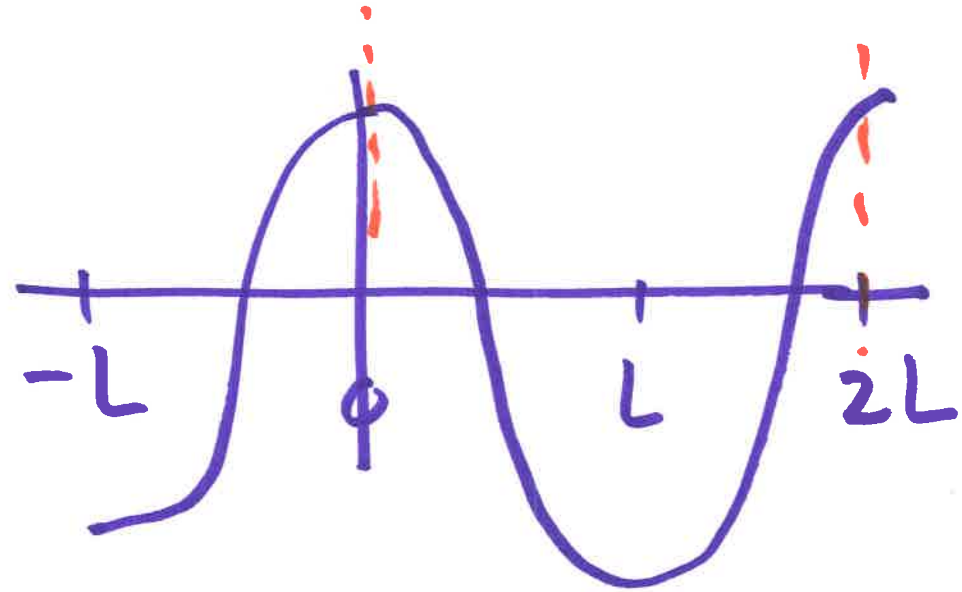
Cartoon



Examples: 1) $\sin\left(\frac{\pi x}{L}\right)$ periodic of length $2L$



2) $\cos\left(\frac{\pi x}{L}\right)$ periodic of length $2L$



3) $\sin\left(\frac{n\pi x}{L}\right)$ periodic of length $\frac{2L}{n}$

$\cos\left(\frac{n\pi x}{L}\right)$

in particular periodic of length $2L$

<u>Return to Fourier sine series:</u>

Suppose $f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

on $[0, L]$

Fourier series extends to <u>$2L$-periodic</u> fn on $\mathbb{R}$. So $f(x)$ should also

In previous exer : $f(x) = 1$ of course extends nicely to all of $\mathbb{R}$

But still have problem

$$f(0) = 1 \neq \text{FS at } x = 0 = 0$$

Reason $f(x) = 1$ even fn

$\sin(nx)$ odd fn

Def: $f : \mathbb{R} \to \mathbb{R}$ is

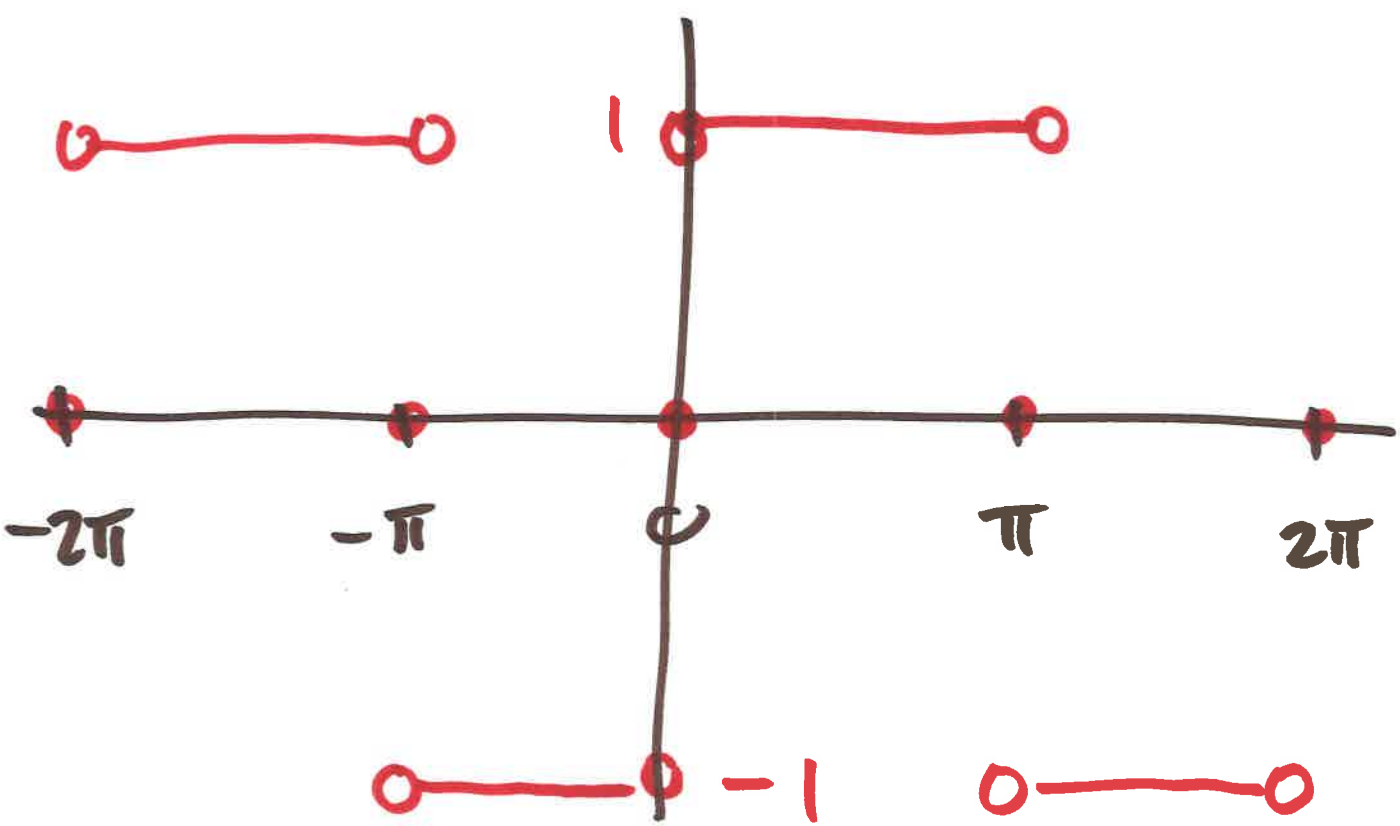
1) even: $f(-x) = f(x)$

2) odd: $f(-x) = -f(x)$

Ex 1) even $\cos\left(\frac{n\pi x}{L}\right)$

2) odd $\sin\left(\frac{n\pi x}{L}\right)$

To extend $f(x) = 1$ on $[0, \pi]$ to be periodic and odd we need to take



Exer Show any $f:\mathbb{R}\to\mathbb{R}$ is uniquely a sum of an even and odd fn

$$f(x) = f_{even}(x) + f_{odd}(x)$$

Suggests: If we can write odd fns as F. sine series and even fns as F cosine series, then we can handle all fns.

For fns on $[-L, L]$, we have beautiful "orthogonal basis"

$$\cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, \ldots$$

$$\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

with respect to inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\, g(x)\, dx$$

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = 0 \qquad \text{all } m,n$$

$$\left\langle \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) = \begin{cases} 0 & m \neq n \\ L & m = n > 0 \\ 2L & m = n = 0 \end{cases}$$

Define Fourier coeffs of $f : [-L, L] \to \mathbb{R}$

$$a_n = \frac{\langle f, \cos(\frac{n\pi x}{L}) \rangle}{\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle}$$

$$b_n = \frac{\langle f, \sin(\frac{n\pi x}{L}) \rangle}{\langle \sin(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L}) \rangle}$$

More explicitly

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad \underline{n=1,2,3,\ldots} \quad > 0$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot 1\, dx \qquad \underline{n=0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \underline{\text{all}}\ n=1,2,\ldots$$

Define Fourier cos and sin series of $f: [-L, L] \to \mathbb{R}$

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Caution Some people, such as the book's authors, define a_0 to be twice our a_0 and then divide by two...

<u>Thm</u> $f : \mathbb{R} \to \mathbb{R}$ periodic of length $2L$ with f' pw cont.

Then

$$\text{Fourier cos and sin series at } x = \begin{cases} f(x) & f \text{ cont at } x \\ \dfrac{f(x^-)+f(x^+)}{2} & \text{in general} \end{cases}$$