\[ \frac{8}{11^2} = 1 + \frac{1}{1 + \frac{1}{25 + \frac{1}{94 + \frac{1}{25 + \frac{1}{94 + \ldots}}}} \]

Final: Wed 12/16 3-6pm Wheeler Hall.

Next week: Review during usual lecture times.

Friday Quiz through 9.10.3.

Today Office Hours 12:30-2pm, 3-5pm.

I became a Mathematician.

Lecture 25: Fourier Series, or How
1) with order Linear ODEs

\[ y^{(m)} + p_{m-1} y^{(m-1)} + \cdots + p_1 y' + p_0 y = f \]

2) Homogeneous gen soln
\[ y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \]

3) Nonhomogeneous gen soln
\[ y = y_h + y_p \]

IVP unique soln with \( y(0) = y', \ldots, y^{(m-1)} \)

\[ y(0) = y' = \cdots = y^{(m-1)} = 0 \]
General Theory

5) Nonhomogeneous: gen soln
\[ y = y_0 + c_1 y_1 + \ldots + c_n y_n \]

2) Systems of 1st order ODEs
\[ y_1' = A y_1 + f \]

Depending on aux eqn and nonhomogen term, can find explicit soln.
a) $y = e^{rt}$, $r \in \text{e-vector}$
b) columns of $A$

techniques to homogenize

find some explicit solutions to

(c) IVP: unique soln with $y(0) = 1$
Relation between 1) nth order ODEs and 2) 1st order systems of ODEs

\[ y_n = \ldots y_2 \ y_1 \ y_0 \]

1. Converts nth order ODEs into 1st order systems of ODEs
2. Forms 1st order systems of ODEs from nth order ODEs
\[
\begin{align*}
\frac{\partial n}{\partial t} &= nL \\
n(0,t) &= n_0 \\
n(L,t) &= n_0 \\
\end{align*}
\]

We also discussed BVP:

\[
\begin{align*}
\text{BVP: } n(0,t) &= 0 = n(L,t) \\
f(x) &= n(x', 0) = f(x) \\
\end{align*}
\]

Back to Heat Energy

\[
\frac{\partial^2 n}{\partial y^2} = \frac{te}{ne}
\]
\[
\begin{align*}
&\frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\
&\frac{\pi}{2} \leq x \leq \pi \\
\end{align*}
\]

\[
\begin{align*}
&f(x) = (x-rac{\pi}{4})^2 \\
&y = L = \frac{\pi}{2} \\
&\text{and} \\
&\text{Return to Exer. Solve Heat Eqn. with}
\end{align*}
\]
By Separation of Variables we found

\[ u(x,t) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} \right) e^{-\beta \left( \frac{n\pi}{L} \right)^2 t} \]

Gen Soln: \[ u(x,t) \]

Solves Heat Eqn and BVP
Need to solve for coefficients.  What we want is:

\[
\sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} \right) = f(x)
\]

Remaining challenge: solve IVP
inner prod \( \langle f, g \rangle = \int_0^l f(x)g(x)dx \)
with \( f(0) = 0 = f(l) \)
\[ \text{vec} \mathbf{sp} \mathbf{v} = Y = \left\{ f : [0, l] \rightarrow \mathbb{R} \right\} \]
form an orthogonal basis

\( u_n(x, 0) = \sin \left( \frac{n\pi x}{l} \right) \) \( n = 1, 2, 3, \ldots \)

Proced as if Fourier Series: Fourier

\[ \text{Microlocally Effective Idea: Fourier} \]
$\sum_{n=1}^{\infty} \left( \left\langle \mathbf{x}_n, \mathbf{x}_n \right\rangle/\left\langle \mathbf{x}_n, \mathbf{x}_n \right\rangle \right) \mathbf{x}_n + \ldots + \left( \left\langle \mathbf{x}_1, \mathbf{x}_1 \right\rangle/\left\langle \mathbf{x}_1, \mathbf{x}_1 \right\rangle \right) \mathbf{x}_1 = \mathbf{v}$

If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is orthogonal basis $V$

Since in an inner product space $V$

$$c_n = \frac{\left\langle \mathbf{f}, \mathbf{x}_n \right\rangle}{\left\langle \mathbf{x}_n, \mathbf{x}_n \right\rangle} \text{ for } n \leq N$$

0 Outcome will be:
Verify orthogonality:

\[
\left\langle \frac{\sin \left( \frac{m \pi x}{L} \right)}{\sin \left( \frac{n \pi x}{L} \right)} \right\rangle \left. \frac{\sin \left( \frac{m \pi x}{L} \right)}{\sin \left( \frac{n \pi x}{L} \right)} \right|_0^L = \frac{1}{L} \int_0^L \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) dx
\]
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right) \]

with
\[ c_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx \]

Thm:
Any function \( f: [0, L] \to R \) with \( f(0) = f(L) \) is equal to its Fourier series.

Confirm they form a "basis" since we allow
(not really a basis since we allow "dirac combs")
\[
f(x) = \sum_{n=1}^{\infty} \text{cn} \sin(nx)
\]

Solve for \( c \).

Back to Exer.
\[
\left( \cdots + \frac{25}{1} \sin(5x) - \frac{6}{1} \sin(x) - \frac{1}{1} \sin(3x) \right) \frac{\pi}{4} = f(x)
\]

This says:

\[
\begin{cases}
\frac{\pi}{4} n & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even}
\end{cases}
\]

Intermediate by parts yields:
Finally soln to Heat Eqn solving BVP and IVP

\[ u(x,t) = \frac{4}{\pi} \left( \sin(x) e^{-t} - \frac{1}{9} \sin(3x)e^{-9t} \right. \\
\left. + \frac{1}{25} \sin(5x)e^{-25t} - \ldots \right) \]
\[
\left(... + \frac{9}{16} + \frac{25}{1} + \frac{1}{1} + \frac{25}{16} + ... \right) = 4 \left(1 + \frac{b}{4} + \frac{b}{1} + \frac{1}{1} + \frac{b}{16} + ... \right)
\]
\[
\frac{25}{2 \pi} \sin \left(\frac{2}{4 \pi} \right) + \frac{1}{1} + \frac{1}{16} \sin \left(\frac{2}{8 \pi} \right) - \frac{b}{16} \sin \left(\frac{2}{8 \pi} \right)
\]

This says

\[
\frac{2}{\pi} f = \frac{2}{\pi} \left(\frac{2}{4 \pi} \right) = f \left(\frac{2}{4 \pi} \right)
\]

Amazing observation in exercise.
Whoa!

\[ \frac{1}{8} = \frac{1}{15} + \frac{1}{6} + \frac{1}{5} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \ldots \]