<u>Lecture 24</u> Solving the <u>Heat</u> <u>Equation</u>!

<u>Today</u>: Office Hours <u>1:30 - 3</u>, <u>736</u> Evans

Please fill out course evaluations!

"The single biggest problem in communication is the illusion that it has taken place."

— George Bernard Shaw

Happy Thanksgiving!

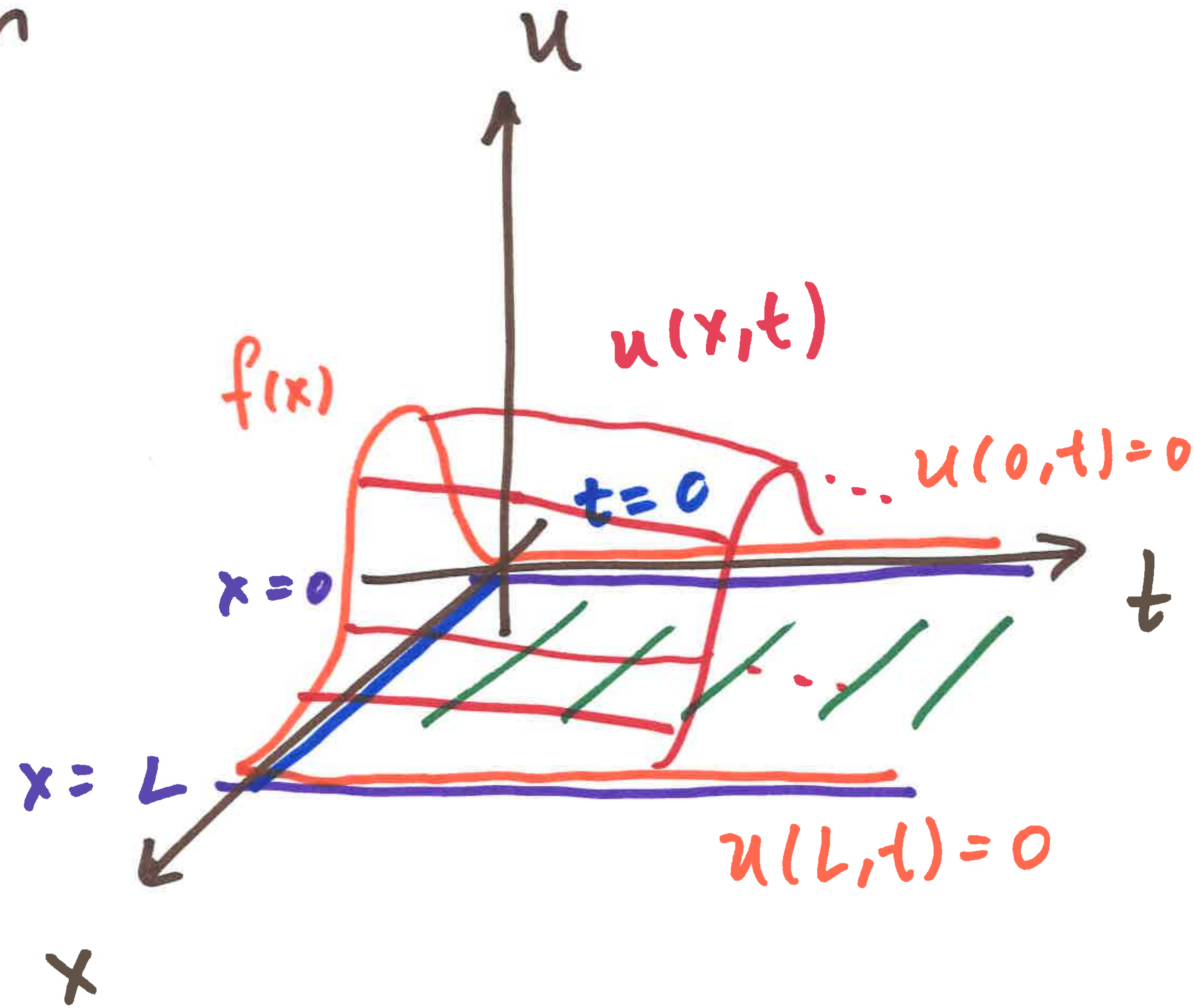
Recall Heat Eqn: $u = u(x,t)$ temp in a rod of length L

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

IVP: $u(x,0) = f(x)$

BVP: $u(0,t) = 0 = u(L,t)$

Cartoon



Method : Separation of Variables

Seek soln of form $u(x,t) = X(x)\,T(t)$

Obtained : Decoupled ODEs

(*) $T'(t) + \lambda\beta\, T(t) = 0$

(**) $X''(x) + \lambda X(x) = 0$

λ constant any λ is possible

but we will see $\lambda > 0$ is only interesting case

For <u>example</u> suppose $\underline{\lambda = 0}$

$$(*) \quad T'(t) = 0 \qquad\qquad T(t) = a$$

$$\Longrightarrow$$

$$(**) \quad X''(x) = 0 \qquad\qquad X(x) = bx + c$$

<u>BVP</u> $\Rightarrow X(0) = 0 = X(L)$

(or $T(t) = 0$ so $\underline{u(x,t) = 0}$)

But $X(x) = bx + c$ so $b = c = 0$

so $X(x) = 0$ so $\underline{u(x,t) = 0}$

Gen solns

(*) $T(t) = a e^{-\lambda\beta t}$

(**) Aux eqn $r^2 + \lambda = 0$

$\lambda > 0$ $X(x) = b_1 e^{i\sqrt{\lambda}x} + b_2 e^{-i\sqrt{\lambda}x}$

or alt $X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

Exer **$\lambda < 0$** : no interesting solns to (**) satisfying BVP

Analyze BVP: If $u(x,t)$ is not identically 0, then need

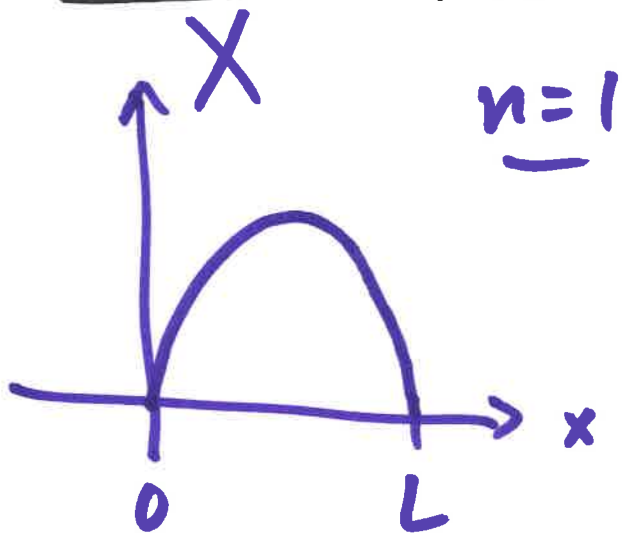
$$X(0) = 0 = X(L)$$

$\underline{X(0)=0} \implies c_1 = 0$ so $X(x) = c_2 \sin(\sqrt{\lambda}\, x)$

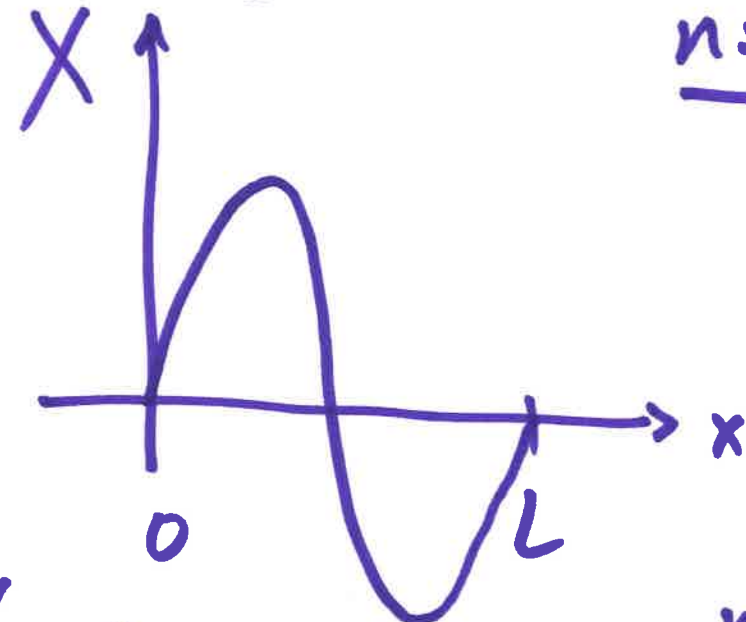
$\underline{X(L)=0} \implies \sqrt{\lambda} = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots$

so $X(x) = c_2 \sin\left(\frac{n\pi}{L} x\right)$

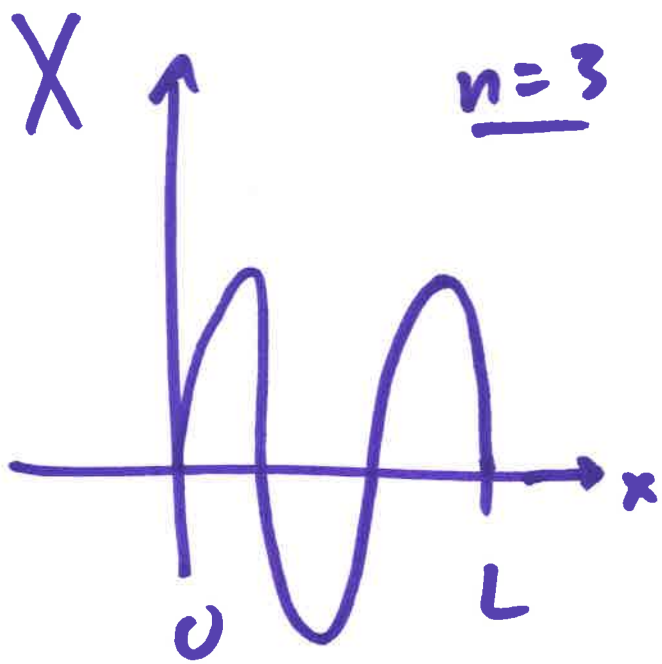
<u>Define</u> $X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \quad n=1,2,3,\ldots$



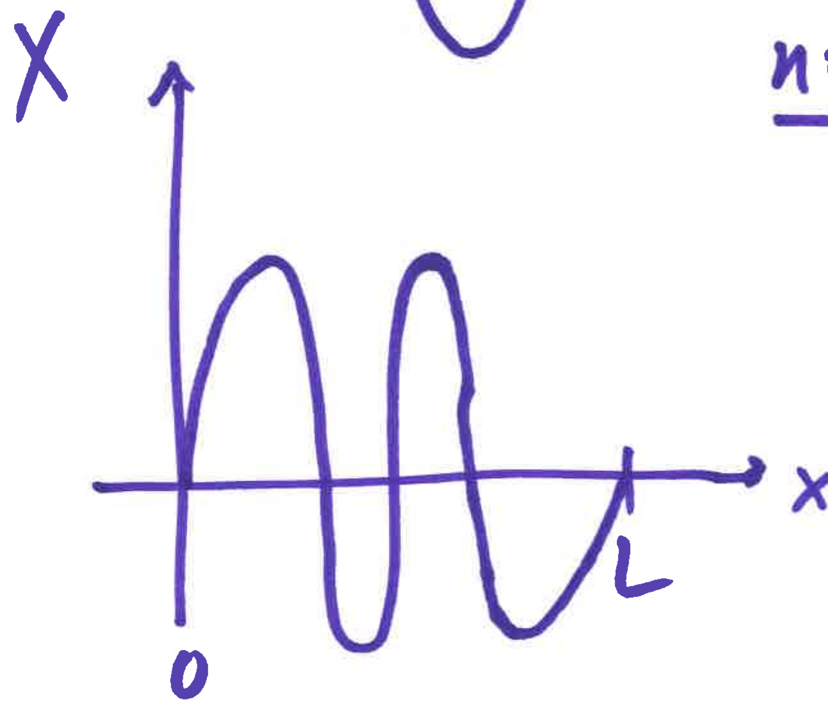
$\underline{n=1}$



$\underline{n=2}$



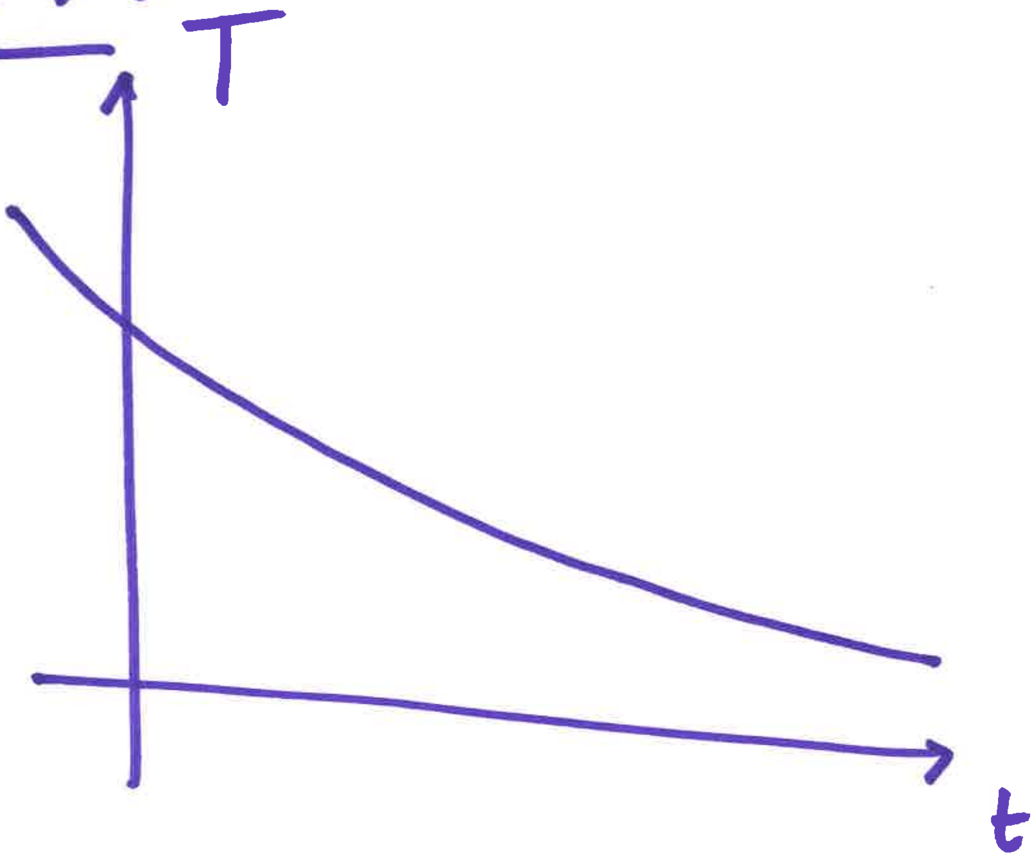
$\underline{n=3}$



$\underline{n=4}$

__Define__ $T_n(t) = e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$ $n = 1, 2, 3, \ldots$

<ins>$\beta > 0$</ins>



Collecting our work: we have list of solns

$$u_n(x,t) = X_n(x)\,T_n(x)$$

$$= \sin\left(\frac{n\pi}{L}x\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

$n = 1, 2, 3, \ldots$

Solve heat eqn

& BVP

<u>Hope</u> this is a "<u>basis</u>" of solns if we allow "<u>∞ - lin combs</u>"

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

(We don't usually allow ∞-li combs or bases to require ∞-lin combs)

Analyze IVP

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0)$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

Need to solve for c_n

<u>Pretend</u> this is a <u>matrix</u> eqn:

$$\begin{bmatrix} | \\ f \\ | \end{bmatrix} = \begin{bmatrix} | & | & | & \\ X_1(x) & X_2(x) & X_3(x) & \cdots \\ | & | & | & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}$$

Want f to be in "∞-span" of $X_1, X_2, X_3, \ldots$

Exer Solve **heat eqn** $\dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2}$

with **IVP**: $u(x,0) = 3\sin(2x) - 6\sin(5x)$

BVP: $u(0,t) = 0 = u(L,t)$

where $L = \pi$, $\beta = 7$

Soln Gen soln $u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-7n^2 t}$

Solves **heat eqn** & **BVP**

Now to solve <u>IVP</u>:

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

<u>Need</u> $= 3\sin(2x) - 6\sin(5x)$

<u>Take</u> $c_2 = 3$, $c_5 = -6$ all other $c_n = 0$.

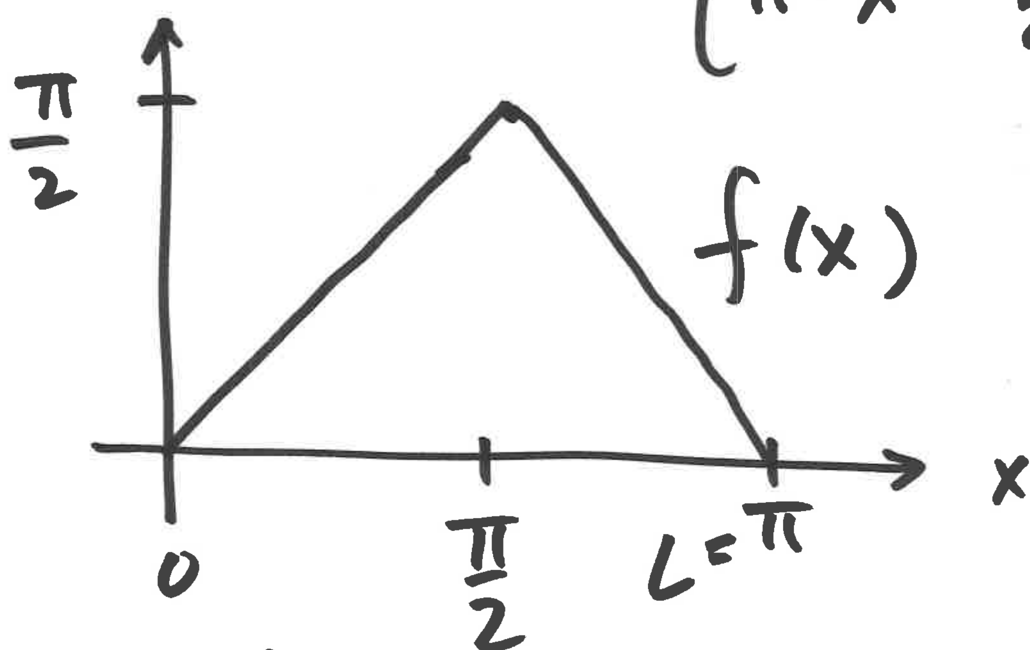
Conclusion we found soln

$$u(x,t) = 3\sin(2x)e^{-7\cdot 4\cdot t} - 6\sin(5x)e^{-7\cdot 25\cdot t} \;!$$

<u>Exer</u> Solve <u>heat eqn</u> $\dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2}$

with <u>BVP</u> : $u(0,t) = 0 = u(L,t)$

<u>IVP</u> : $u(x,0) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$



$L = \pi$, $\beta = 1$

__Soln__ Gen soln

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

__IVP__ $$u(x,0) = \sum_{n=1}^{\infty} c_n u_n(x,0)$$

__Want__

$$= \sum_{n=1}^{\infty} c_n \sin(nx) = f(x)$$

Need to solve for c_n's

Miraculously Effective Idea:

Proceed as if

$$u_n(x,0) = \sin\left(\frac{n\pi}{L}x\right) \quad n=1,2,3,\dots$$

form an "orthogonal basis"

for vect sp $V = \{ \text{fns } f:[0,L] \to \mathbb{R} \text{ so that } f(0)=0,\ f(L)=0 \}$

inner prod $\langle f,g \rangle = \int_0^L f(x)g(x)\,dx$

<u>Outcome will be</u>:

$$c_n = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle}$$

<u>Conclude</u>: $u(x,t) = \sum_{n=1}^{\infty} \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n(x,t)$

solves heat eqn, BVP & IVP with $u(x,0) = f(x)$