Lecture 20 Nonhomogeneous 2nd Order Diff Eqns

Today Abbreviated Off Hrs 1-2 pm
736 Evans

Friday: Quiz through §4.5

Warmup Solve IVP

$$y'' - 4y' + 5y = 0$$
$$y(0) = 3$$
$$y'(0) = -4$$

Soln Aux eqn $r^2 - 4r + 5 = 0$

Quad eqn $\rightsquigarrow$ roots $= 2 \pm i$

Basis of solns $e^{(2+i)t}, e^{(2-i)t}$

Alternative real basis

$$\frac{e^{(2+i)t} + e^{(2-i)t}}{2} = e^{2t}\cos(t)$$

$$\frac{e^{(2+i)t} - e^{(2-i)t}}{2i} = e^{2t}\sin(t)$$

__Gen soln__ $y = c_1 e^{2t}\cos(t) + c_2 e^{2t}\sin(t)$

			__Want__
$y(0)$	$=$	c_1	$= 3$
$y'(0)$	$=$	$2c_1 + c_2$	$= -4$

__Solve lin syst:__

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Find $c_1 = 3$, $c_2 = -10$

Soln to IVP

$$y = 3e^{2t}\cos(t) + (-10)e^{2t}\sin(t)$$

__Thm__ I) Can always find basis for solns of $y'' + by' + cy = 0$

Solns take following form depending on roots of aux eqn $r^2 + br + c = 0$

1) $\underline{r_1 \neq r_2 \text{ real}}$: $e^{r_1 t}, e^{r_2 t}$

2) $\underline{r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta \text{ complex}}$ $e^{r_1 t}, e^{r_2 t}$

or alternatively $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$

3) $r_1 = r_2$ repeated real: e^{rt}, te^{rt}

(check this!)

II) (Wronskian Lemma) If y_1, y_2 are lin indep solns, then

$$\begin{bmatrix} y_1(0) \\ y_1'(0) \end{bmatrix}, \begin{bmatrix} y_2(0) \\ y_2'(0) \end{bmatrix}$$ are also lin indep

So can always uniquely solve IVP

$$y(0) = Y_0, \quad y'(0) = Y_1$$

Since we only need to solve lin syst

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

Wronskian Lemma confirms matrix is invertible

Now on to nonhomog. eqns!

$$y'' + by' + cy = f$$

Lin alg interpretation: V vect sp of fns on $\mathbb{R}$

$T : V \longrightarrow V$ lin transf

$$T = \left(\frac{d}{dt}\right)^2 + b\frac{d}{dt} + cI$$

$$\left.\begin{aligned} \underline{b} &= f \\ \underline{x} &= y \end{aligned}\right\} \text{vectors}$$

Solve

$$T\underline{x} = \underline{b}$$

Method? If T were matrix, we would row reduce ...

Next best idea? Educated guess!
(method of undetermined coeffs)

Exer Find a soln $y'' + 2y' + y = t^2$

Soln
$$\left.\begin{array}{l} 1 \xmapsto{T} 1 \\ t \longmapsto 2+t \\ t^2 \longmapsto 2+4t+t^2 \end{array}\right\}$$

t^2 is in span of

Try $y = A_2 t^2 + A_1 t + A_0$ and solve for A_2, A_1, A_0

Comes down to lin syst:

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \\ T(t^2) \quad T(t) \quad T(1) \end{array} \begin{array}{c} \begin{bmatrix} A_2 \\ A_1 \\ A_0 \end{bmatrix} \\ \end{array} \begin{array}{c} = \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ t^2 \end{array} \qquad \begin{array}{c} \underline{\text{basis}} \\ t^2 \\ t \\ 1 \end{array}$$

Find $A_2 = 1$, $A_1 = -4$, $A_0 = 6$

So $y = t^2 - 4t + 6$ is

a soln to $y'' + 2y' + y = t^2$!

<u>Thm</u> To find a soln to

$$y'' + by' + cy = t^m$$

we can guess

$$y = A_m t^m + \cdots + A_1 t + A_0$$

and solve lin syst for

$$A_m, \ldots, A_1, A_0$$

Lin syst takes the form

$$\begin{bmatrix} | & | & & | \\ T(t^m) & T(t^{m-1}) & \cdots & T(1) \\ | & | & & | \end{bmatrix} \begin{bmatrix} A_m \\ \vdots \\ A_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$T = \left(\frac{d}{dt}\right)^2 + b\left(\frac{d}{dt}\right) + cI$$

Exer Find a soln to

$$y'' + 3y' + 2y = e^{3t}$$

Solns (Basis for homog eqn solns

$$e^{-t}, e^{-2t} \;)$$

Guess $e^{3t} \xmapsto{T} 20e^{3t}$

Take $y = \dfrac{e^{3t}}{20}$!

Exer Find a soln to

$$y'' + 3y' + 2y = e^{-2t}$$

soln to homog eqn!

Soln Guess

$$e^{-2t} \xmapsto{T} 0$$

$$te^{-2t} \longmapsto -e^{-2t}$$

Take $y = -te^{-2t}$!

<u>Thm</u> We can always solve

$$y'' + by' + cy = e^{rt}$$

Solns depend on roots of aux eqn

1) r <u>not a root</u> : $y = de^{rt}$

2) r <u>simple root</u> : $y = dte^{rt}$

3) r <u>repeated root</u> : $y = dt^2e^{rt}$

In above d is a constant to solve for.

Exer Find a soln to

$$y'' - 2y' + y = te^t$$

soln to homog eqn

Soln (Basis for homog eqn solns

$$e^t, \; te^t$$)

Guess

$$t^2e^t \stackrel{T}{\longmapsto} 2e^t$$

$$t^3e^t \longmapsto 6te^t$$

Take $y = \dfrac{t^3e^t}{6}$!

Thm We can always solve

$$y'' + by' + cy = t^m e^{rt}$$

Solns depend on roots of aux eqn

1) r not a root: $y = (A_m t^m + \dots + A_0)e^{rt}$

2) r simple root: $y = t(A_m t^m + \dots + A_0)e^{rt}$

3) r repeated root: $y = t^2(A_m t^m + \dots + A_0)e^{rt}$

In above $A_m, \dots, A_0$ are constants to solve for

__Remarks__ 1) There is a similar thm for $t^m e^{\alpha t}\cos(\beta t)$, $t^m e^{\alpha t}\sin(\beta t)$

(can be derived from thms we discussed if we allow complex numbers

2) __Superposition principle:__ If right hand side is $f = f_1 + f_2$, then soln will be of form $y = y_1 + y_2$ where y_1 solves for f_1, y_2 solves for f_2