Lecture 2: Solving Lin. Systems!

Today: Office Hours 1-3 pm
736 Evans

Friday: Quiz through § 1.3

Welcome Back!

1015 Evans

MUSA pizza party 9/2, 6-8 pm

Last time: A lin syst.

$$\underset{\text{eqns}}{m=2}\left\{\underbrace{\begin{aligned} 2x_1 - 3x_2 \qquad\quad - 5x_4 &= 7 \\ 2x_2 + 7x_3 - 2x_4 &= 0 \end{aligned}}_{n=4 \text{ vars}}\right.$$

can be encoded by an avg. matrix

$$\left[\begin{array}{cccc|c} 2 & -3 & 0 & -5 & 7 \\ 0 & 2 & 7 & -2 & 0 \end{array}\right]$$

__Now__: Solve lin systs!

__Strategy__: change to an __equivalent__ but __simpler__ lin syst.

__Using__: (R~~3~~1) Add mult. of __any__ row to any other row

(R2) Exchange rows

(R3) Scale a row by nonzero number

Goal Row Echelon Form (REF)

$$
\left[\begin{array}{ccccccccccc|c}
0 & 0 & \cdots & 0 & \blacksquare & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \blacksquare & * & \cdots & \cdots & * \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \blacksquare & * & \cdots & * \\
\vdots & & & & & & & & & & & \vdots \\
0 & \cdots & \cdots & & & & & & & & 0 & \blacksquare \\
0 & \cdots & \cdots & & & & & & & \cdots & \cdots & 0
\end{array}\right]
$$

$\blacksquare \neq 0 \longleftarrow$ pivot / leading entries

$*$ = any number

Exer

$$\begin{bmatrix} 2 & 0 & 1 & -1 & 5 \\ 0 & 3 & 0 & 0 & c \\ 0 & d & -2 & 0 & 1 \end{bmatrix}$$

For what c, d is this in REF?

$d = 0$, c any number

Even better Reduced Row Ech. Form (RREF)

$$
\begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & 0 & * & \cdots & * & 0 \\
0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & 0 & * & \cdots & * & \vdots \\
0 & \cdots & \cdots & \cdots & & & & & 0 & 1 & * & \cdots & * & \vdots \\
\vdots & & & & & & & & & & & & \vdots & \\
 & & & & & & & & & & & & * & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & & & & & & & & 0 & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & & & & & & & & & 0
\end{bmatrix}
$$

▨ became = 1

above pivots became = 0.

Exer For what c, d is the following in RREF?

$$\begin{bmatrix} 0 & c & 2 \\ 0 & d & +1 \end{bmatrix}$$

None!

<u>Soln set</u> <u>free vars</u> x_2, x_4 any <u>numbers</u>

then <u>solve for</u> <u>pivot vars</u> x_1, x_3

$$x_1 = 3 + 2x_2 + x_4$$

$$x_3 = -2x_4$$

Easy to solve lin syst whose aug matrix is in RREF!

$$\begin{aligned} x_1 - 2x_2 \quad\quad - x_4 &= 3 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

x_1, x_3 <u>pivot vars</u>

others: x_2, x_4 <u>free vars</u>

Caution

$$x_1 + 2x_2 - x_3 = 0$$

$$0 = 1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Lin syst will be inconsistent

$\Longleftrightarrow$

there is a pivot entry in aug. col.

Theorem Given any (augmented) matrix it is possible to find a RREF matrix equivalent to the original matrix by row operations. In fact the RREF is unique.

The utility of this statement is that the proof is an *algorithm*.

Algorithm for RREF

Step 1 Find nonzero leftmost column

Step 2 Find nonzero entry of this column
Exchange rows (R2) to move entry to top row

Step 3 Use (R1) to create zeros below this entry

Step 4 Repeat steps 1, 2, 3 with submatrix

Step 5 Scale to make pivots = 1 + Create zeros above pivots

$$\begin{bmatrix} 0 & 0 & 0 & -1 & -2 \\ 0 & \boxed{2} & -2 & -2 & -1 \\ 0 & 2 & -2 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R2} \begin{bmatrix} 0 & \boxed{2} & -2 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 2 & -2 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R1} \begin{bmatrix} 0 & \boxed{2} & -2 & -2 & -1 \\ 0 & 0 & 0 & \boxed{-1} & -2 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix}$$

$$\xrightarrow{R1} \begin{bmatrix} 0 & 2 & -2 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{REF}$$

$$\xrightarrow{R3} \begin{bmatrix} 0 & 1 & -1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1} \begin{bmatrix} 0 & 1 & -1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

__Exer__ For what c, d is below lin sys consistent, what then are its solns?

$$x_1 - 3x_2 = 1$$
$$2x_1 + cx_2 = d$$

$$\left[\begin{array}{cc:c} 1 & -3 & 1 \\ 2 & c & d \end{array}\right]$$

Apply algorithm.

$$\xrightarrow{R1} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & c+6 & d-2 \end{array}\right]$$

REF for any c, d

Cases:

1) $c=-6, d \neq 2$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & d-2 \neq 0 \end{array}\right]$$

Inconsistent!

2) $\underline{c \neq -6, \text{ any } d}$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & \begin{matrix} c+6 \\ \neq 0 \end{matrix} & d-2 \end{array}\right]$$

x_1, x_2 pivot vars

(no free vars)

Soln set: $\left\{ \begin{array}{l} x_2 = \dfrac{d-2}{c+6} \\ x_1 = 1 + 3\left(\dfrac{d-2}{c+6}\right) \end{array} \right.$

3) $\underline{c = -6, d = 2}$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

x_1 pivot var

x_2 free var

Soln set: any x_2, $x_1 = 1 + 3x_2$

__Next topic__: writing lin systs in terms of __col. vectors__.

__Def__ An __n-vector__ is an ordered list of n numbers, drawn as an $n \times 1$ matrix

$$\underline{u} = \underbrace{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right\}n}_{1}$$

a_i's numbers

$\underline{u}$ n-vector

3_ Equivalent presentations of lin syst:

1) Lin syst.
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

2) Augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 7 \\ 1 & 0 & -2 & 0 \end{array}\right]$$

3) Matrix equation

$$\begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$