# Lecture 18: More Applications of Orthogonality

Today Office Hours 1-3pm 736 Evans

Friday: Quiz through §7.1

We're in the homestretch!

__Warmup__ Find orthog basis for soln set of $x_1 + 2x_2 + x_3 + 2x_4 = 0$

__Soln__ __Step 1__ Find basis

__Step 2__ Orthogonalize basis ~~[illegible]~~ using Gram-Schmidt

__Step 1__ $A\underline{x} = \underline{0}$ $A = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

free vars

Basis for soln set:

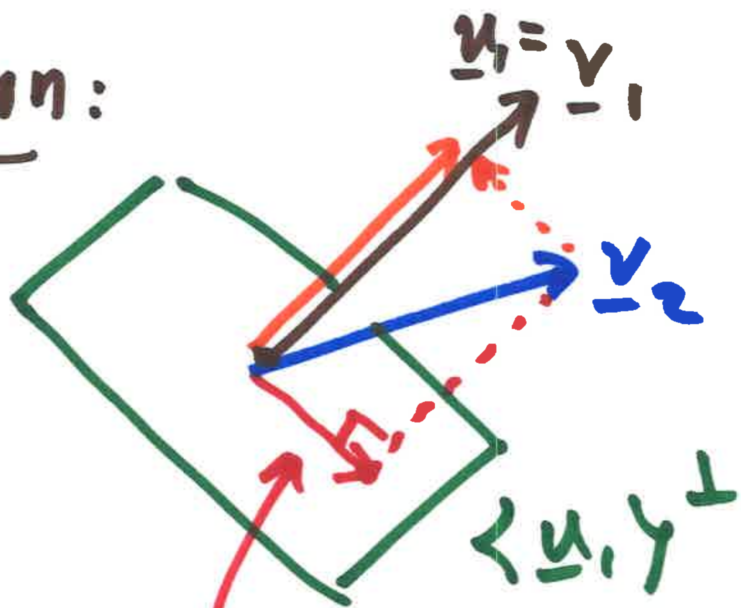
$$\underline{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 2 Apply G-S process

$$\underline{u}_1 = \underline{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{u}_2 = \text{proj}_{\langle \underline{u}_1 \rangle^\perp}(\underline{v}_2) = \underline{v}_2 - \underbrace{\frac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\,\underline{u}_1}_{\text{orange vector in cartoon}}$$

Cartoon:




$\underline{u}_2 = \text{proj}_{\langle \underline{u}_1 \rangle^\perp}(\underline{v}_2)$

$$\underline{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{u}_3 = \mathrm{proj}_{\{\underline{u}_1, \underline{u}_2\}^\perp}(\underline{v}_3)$$

$$= \underline{v}_3 - \frac{\underline{v}_3 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1 - \frac{\underline{v}_3 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\underline{u}_2$$

$$= \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\frac{2}{5}}{\frac{6}{5}}\begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{bmatrix}$$

simplify if you like ...

<u>Exer</u> Find orthog basis for image of $A : \mathbb{R}^2 \to \mathbb{R}^3$, $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

<u>Soln</u> <u>Step</u> 1 Find basis

Use pivot cols $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

<u>Step</u> 2 Apply G-S.

$$\underline{u}_1 = \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{u}_2 = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \cdot \underline{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} - \frac{-2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Organize answer in terms of $A = QR$ factorization

General story: $A = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$ original basis

$Q = \begin{bmatrix} | & & | \\ \hat{\underline{u}}_1 & \cdots & \hat{\underline{u}}_n \\ | & & | \end{bmatrix}$ orthonormal basis

$$\hat{\underline{u}}_i = \underline{u}_i / \|\underline{u}_i\|$$

$A = QR$ where $R$ = matrix relating bases with coeffs from G-S process

**Back to example:**

$$A = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | \\ \hat{u}_1 & \hat{u}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{3} \cdot \frac{1}{\sqrt{\frac{2}{3}}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{3} \cdot \frac{1}{\sqrt{\frac{2}{3}}} \\ \frac{1}{\sqrt{3}} & \frac{2}{3} \cdot \frac{1}{\sqrt{\frac{2}{3}}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{3} \frac{1}{\sqrt{\frac{2}{3}}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{3} \frac{1}{\sqrt{\frac{2}{3}}} \\ \frac{1}{\sqrt{3}} & \frac{2}{3} \frac{1}{\sqrt{\frac{2}{3}}} \end{bmatrix} \underbrace{\begin{bmatrix} \quad R \quad \end{bmatrix}}_{\substack{2 \times 2 \\ \text{upper}\Delta\text{-ar}}}$$

$\|\underline{u}_1\| = \|\underline{v}_1\|$ $\qquad$ (*)

$$R = \begin{bmatrix} \sqrt{3} & \sqrt{3}\cdot\left(-\frac{2}{3}\right) \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

$\|\underline{u}_2\|$

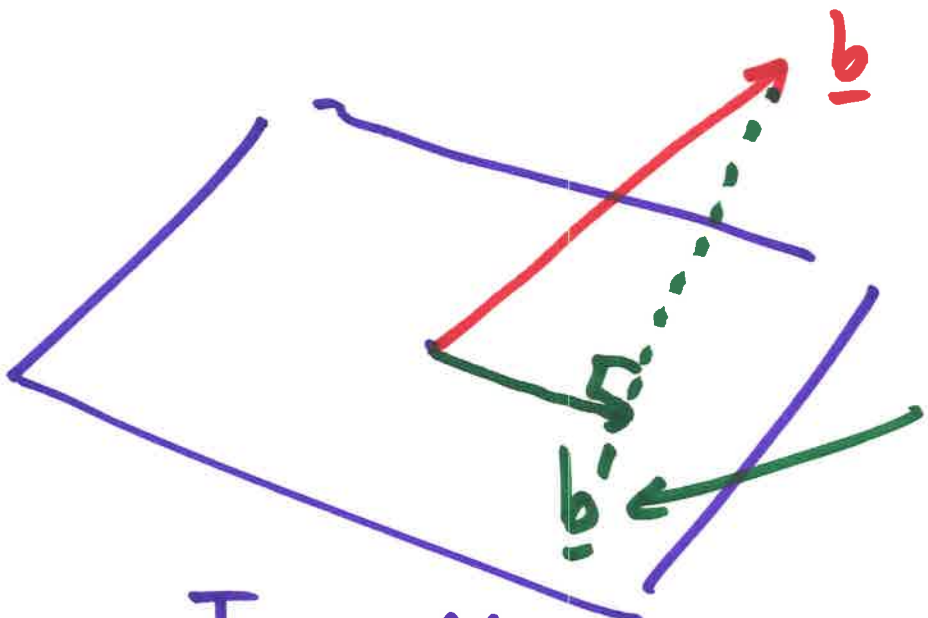
$$\underline{v}_2 = \underline{u}_2 + \frac{\underline{v}_2\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\cdot\underline{u}_1$$

$$= \underbrace{\|\underline{u}_2\|}\,\hat{\underline{u}}_2 + \underbrace{\|\underline{u}_1\|\cdot\frac{\underline{v}_2\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}}_{(*)}\,\hat{\underline{u}}_1$$

# Least Squares Approx

$A\underline{x} = \underline{b}$ linear system

What can we do if inconsistent?



Now solve $A\underline{x}' = \underline{b}'$

"$\underline{x}'$ is vector such that $A\underline{x}'$ is as close to $\underline{b}$ as possible"

We could follow strategy step by step... instead we'll develop simple technique to solve all at one time.

<u>Observe</u>"1) $(\underline{b} - \underline{b}') \perp \mathrm{Image}(A)$

2) $\mathrm{Image}(A) = \mathrm{Col}(A) = \mathrm{Row}(A^T)$

1) & 2) $\Rightarrow$ $\underline{b} - \underline{b}' \in \mathrm{Null}(A^T)$

Suppose we solve $A\underline{x}' = \underline{b}'$. Then

$$A^T(\underline{b} - A\underline{x}') = \underline{0}$$

Rearrange eqn:

$$\underbrace{(A^T A)}_{C}\,\underbrace{\underline{x}'}_{\underline{y}} = \underbrace{A^T \underline{b}}_{\underline{c}}$$

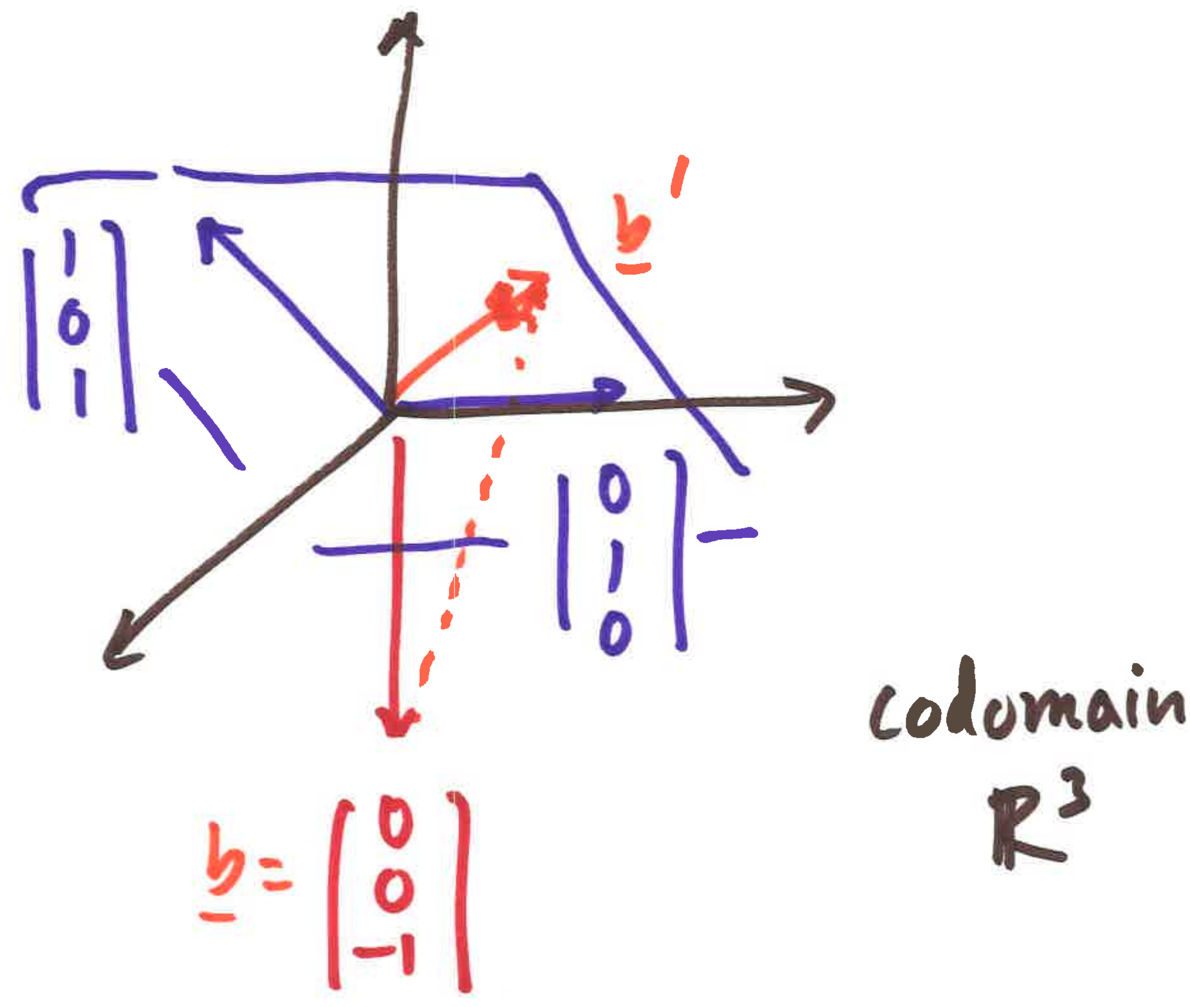
We want to solve

$$C\,\underline{y} = \underline{c}$$

This is a usual lin syst!

Exer  Find least squares approx soln to $\underline{A}\underline{x} = \underline{b}$  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Cartoon



Soln $A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^T \underline{b} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Solve $A^T A \underline{y} = A^T b$

$$\underline{y} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

Aside: $\underline{b}' = A\underline{y}$

$$= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

Another app of orthogonality:

**Spectral Thm** $A$ $n \times n$ matrix has an orthog basic of e-vectors

$$\Longleftrightarrow$$

$A$ is symmetric: $A = A^T$

Rmk In particular, $A$ is diagonalizable

Example: $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ already diag.

coord vectors $\underline{e}_1, \dots, \underline{e}_n$ form orthog basis of e-vectors

and $A = A^T$

Exer Find orthog basis of e-vectors for $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

Soln Note $A = A^T$ so such a basis will exist

$$\chi_A(t) = (1-t)(-1-t) - 3$$
$$= t^2 - 4 = (t-2)(t+2)$$

$\lambda = \pm 2$

$\underline{\lambda = 2}$ $\quad A - 2I = \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix}$

$$\underline{v}_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

$\underline{\lambda = -2}$ $\quad A + 2I = \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

$$\underline{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

<u>Note</u>: $\underline{v}_1 \perp \underline{v}_2$ as required