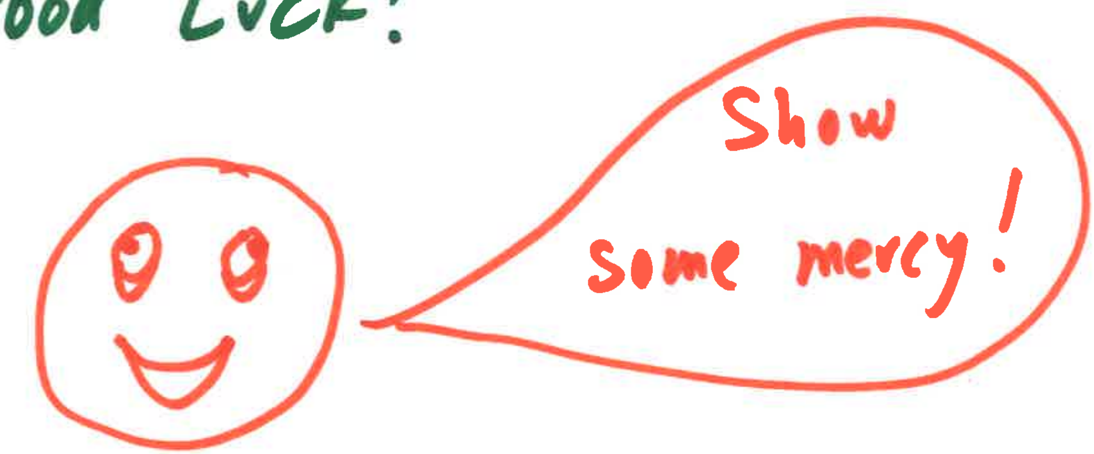
Lecture 17 Orthogonality!

Today: Review session 12:30 - 2pm
2040 VLSB

Offiie hours 2-3pm — 2040 VLSB if available
\ else 736 Evans

Thursday: Midterm 2 through §6.3
Good Luck!

No quiz Friday



<u>Warmup</u> Show $n \times n$ matrix U satisfies

a) $(U\underline{v}) \cdot (U\underline{w}) = \underline{v} \cdot \underline{w}$

("U preserves dot product")

$\Longleftrightarrow$ b) cols of U are orthonormal basis

$\Longleftrightarrow$ c) rows of U are orthonormal basis

$\Longleftrightarrow$ d) U invertible, $U^{-1} = U^T$

Def U satisfying these conditions is called an **orthogonal matrix**

⚠2 Unfortunate terminology... better to have said **orthonormal matrix**

Ex 1) $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation by $\frac{\pi}{2}$

2) $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ reflection across x,y plane

<u>Soln</u> $\mathcal{U} = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}$

<u>a) $\Rightarrow$ b)</u> $\underline{u}_i \cdot \underline{u}_j = (\mathcal{U}\underline{e}_i)\cdot(\mathcal{U}\underline{e}_j)$

$$\overset{a)}{=} \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

<u>b) $\Rightarrow$ d)</u> cols of (o.n.) basis

$\Longrightarrow$ $\mathcal{U}$ is invertible

$$\mathcal{U}^T\mathcal{U} = \begin{bmatrix} - & \underline{u}_1 & - \\ & \vdots & \\ - & \underline{u}_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{bmatrix}$$

so $\mathcal{U}^T = \mathcal{U}^{-1}$

d) $\Rightarrow$ a) $(U\underline{v}) \cdot (U\underline{w}) = (U\underline{v})^T (U\underline{w})$

$$= \underline{v}^T U^T U \underline{w} \overset{d)}{=} \underline{v}^T \underline{w} = \underline{v} \cdot \underline{w}$$

<u>Exercise</u> Show c) is also equivalent to a), b) & d)

Recall from last time: $\underline{v}_1, \ldots, \underline{v}_n$ orthog. basis

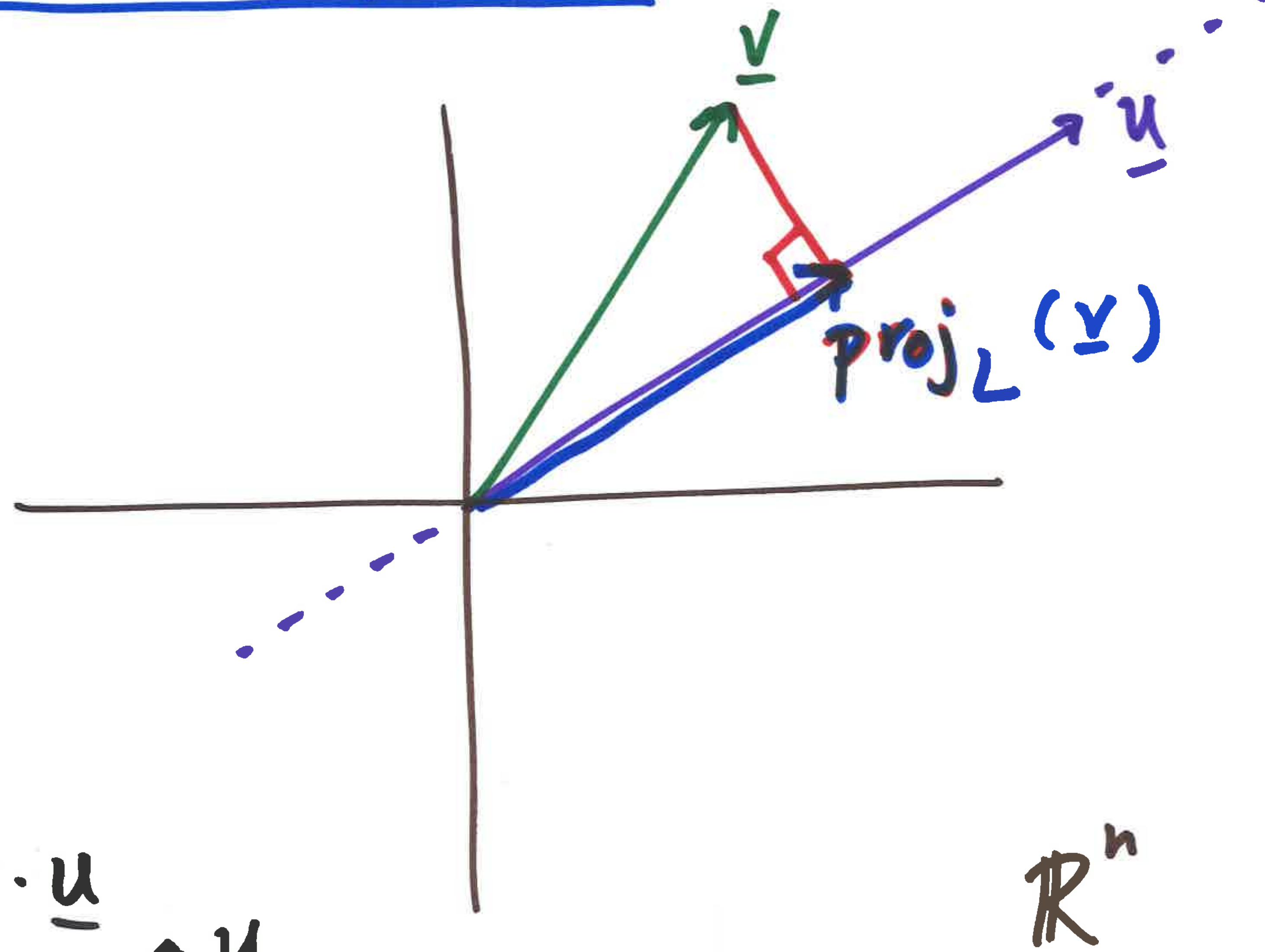
then

$$\underline{v} = \underbrace{\frac{\underline{v} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}}_{a_1} \underline{v}_1 + \cdots + \underbrace{\frac{\underline{v} \cdot \underline{v}_n}{\underline{v}_n \cdot \underline{v}_n}}_{a_n} \underline{v}_n$$

Coords of $\underline{v}$ wrt $\underline{v}_1, \ldots, \underline{v}_n$ have simple formula!

Geometric Interpretation

$L = \text{span}\{\underline{u}\}$

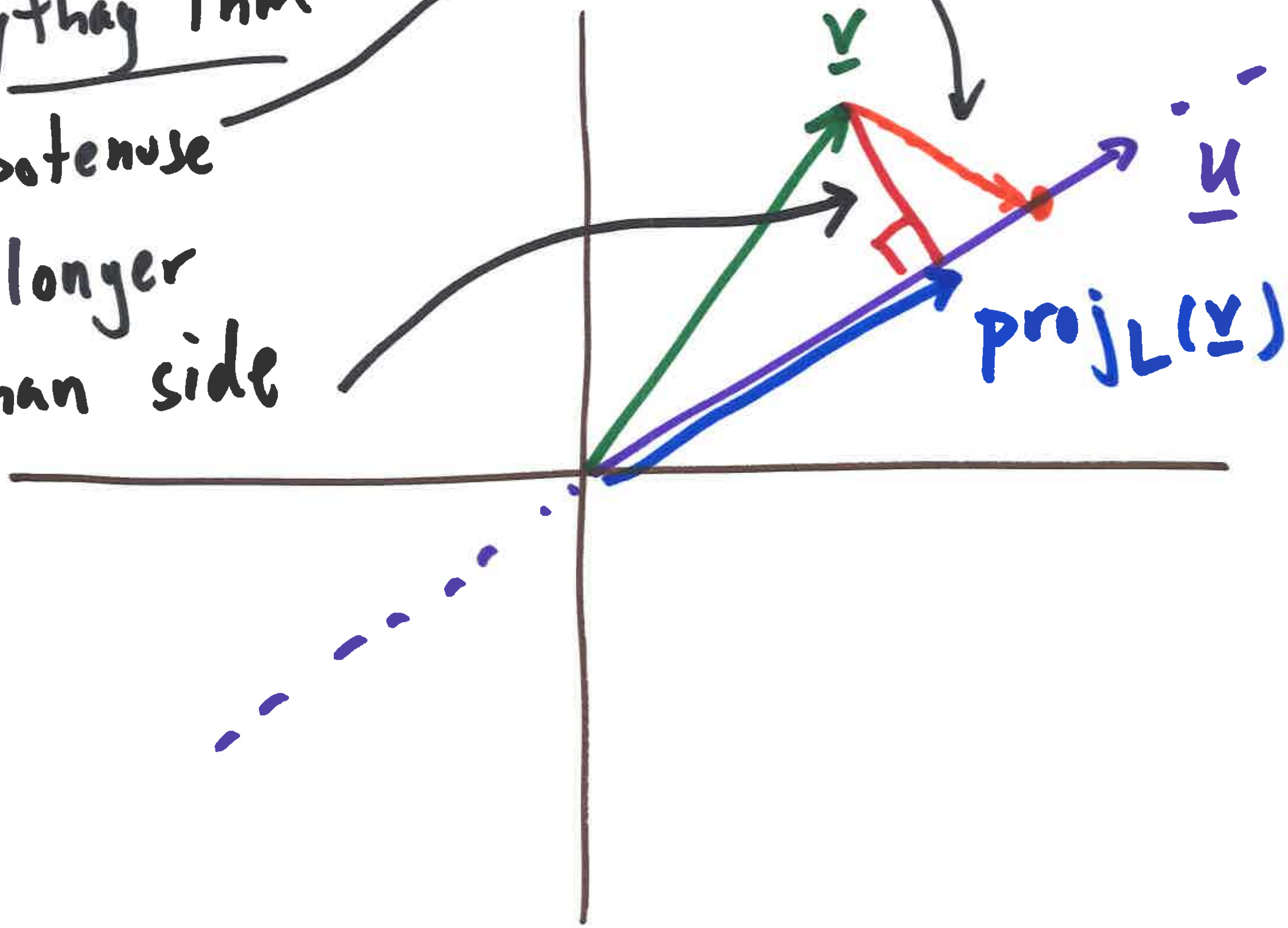


$$\text{proj}_L(\underline{v}) = \frac{\underline{v} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \cdot \underline{u}$$

Exer $\mathrm{proj}_L(\underline{v})$ is closest vector in L to $\underline{v}$

Pythag Thm

hypotenuse is longer than side



<u>Thm</u> (Projections <u>to subspaces</u>)

Suppose $W \subset \mathbb{R}^n$ is subspace with orthogonal $W^\perp \subset \mathbb{R}^n$

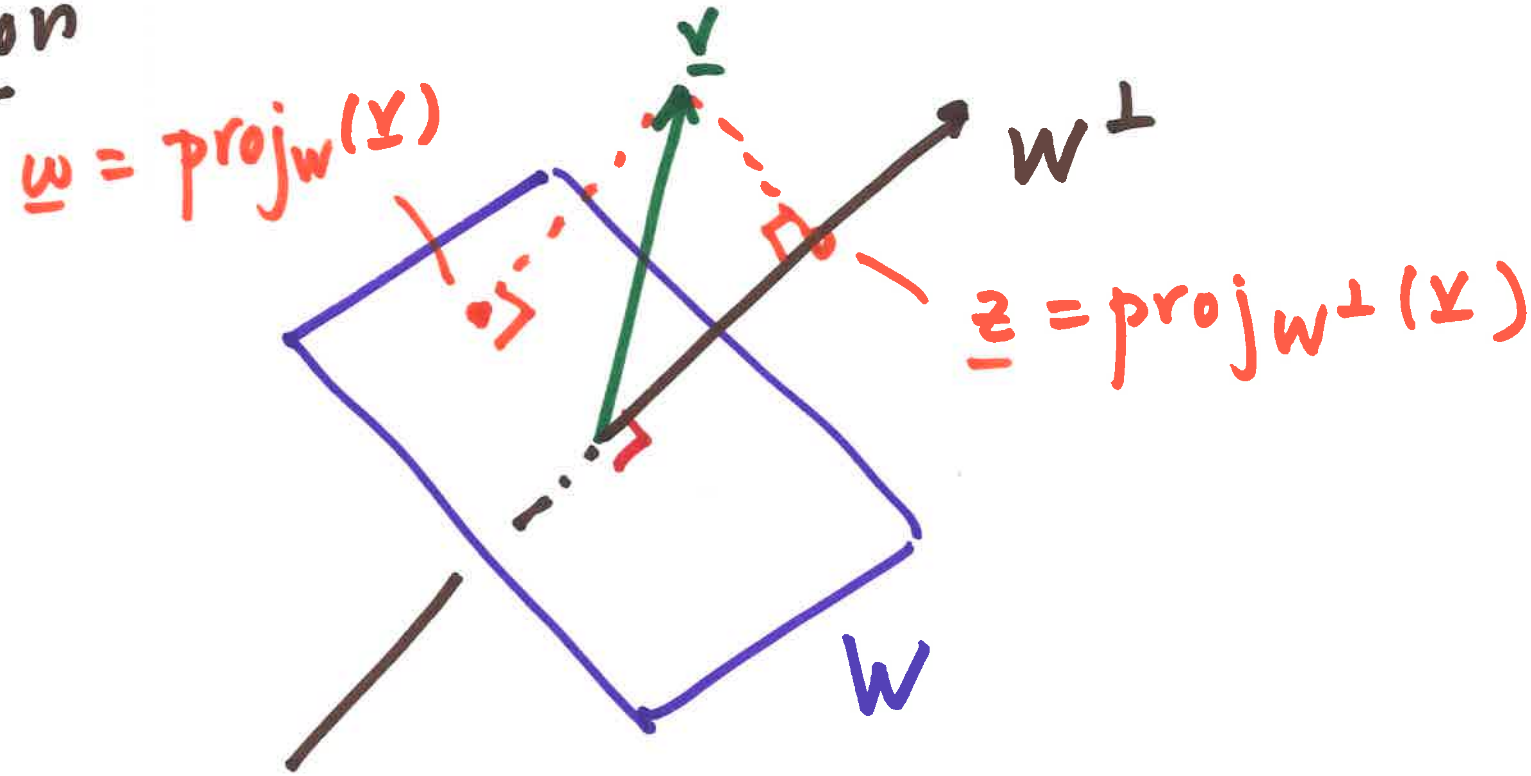
Then any $\underline{v} \in \mathbb{R}^n$ has a <u>unique</u> de<u>comp</u>o<u>sition</u>

$$\underline{v} = \underbrace{\underline{w}}_{\in W} + \underbrace{\underline{z}}_{\in W^\perp}$$

$\operatorname{proj}_W(\underline{v})$

$\operatorname{proj}_{W^\perp}(\underline{v})$

Cartoon



Formula suppose $\underline{w}_1, \ldots, \underline{w}_k$ orthog basis of W

then $\underline{w} = \text{proj}_W(\underline{v}) = \dfrac{\underline{v}\cdot\underline{w}_1}{\underline{w}_1\cdot\underline{w}_1}\underline{w}_1 + \cdots + \dfrac{\underline{v}\cdot\underline{w}_k}{\underline{w}_k\cdot\underline{w}_k}\underline{w}_k$

__Proof__ Take $\underline{w} = \text{proj}_W(\underline{v}) = \frac{\underline{v}\cdot\underline{w}_1}{\underline{w}_1\cdot\underline{w}_1}\underline{w}_1 + \cdots \frac{\underline{v}\cdot\underline{w}_k}{\underline{w}_k\cdot\underline{w}_k}\underline{w}_k$

as in formula

Take $\underline{z} = \text{proj}_{W^\perp}(\underline{v}) = \underline{v} - \underline{w}$

__Observe:__ 1) $\underline{v} = \underline{w} + \underline{z}$ ✓

2) $\underline{w} \in W = \text{span}\{\underline{w}_1, \ldots, \underline{w}_k\}$ ✓

3) __Exer__ check $\underline{z} \in W^\perp$ by

showing $\underline{z}\cdot\underline{w}_i = 0$ all i

Conclusion we can express

$$\underset{\substack{\cup\!\!\mid \\ W}}{\underline{v} = \underline{w}} + \underset{\substack{\cup\!\!\mid \\ W^{\perp}}}{\underline{z}}$$

We need to show it is unique.

Suppose $\underline{v} = \underset{\substack{\cup\!\!\mid \\ W}}{\underline{w}'} + \underset{\substack{\cup\!\!\mid \\ W^{\perp}}}{\underline{z}'}$

Then $\underline{0} = \underset{\substack{\cup\!\!\mid \\ W}}{(\underline{w} - \underline{w}')} + \underset{\substack{\cup\!\!\mid \\ W^{\perp}}}{(\underline{z} - \underline{z}')}$

So $\underset{\substack{\rotatebox{90}{$\in$} \\ W}}{(\underline{w}-\underline{w}')} = -\underset{\substack{\rotatebox{90}{$\in$} \\ W^\perp}}{(\underline{z}-\underline{z}')}$

But $W \cap W^\perp = \{\underline{0}\}$

since $\underline{u} \in W \cap W^\perp$

then $\underline{u}\cdot\underline{u} = 0$ so $\underline{u} = \underline{0}$.

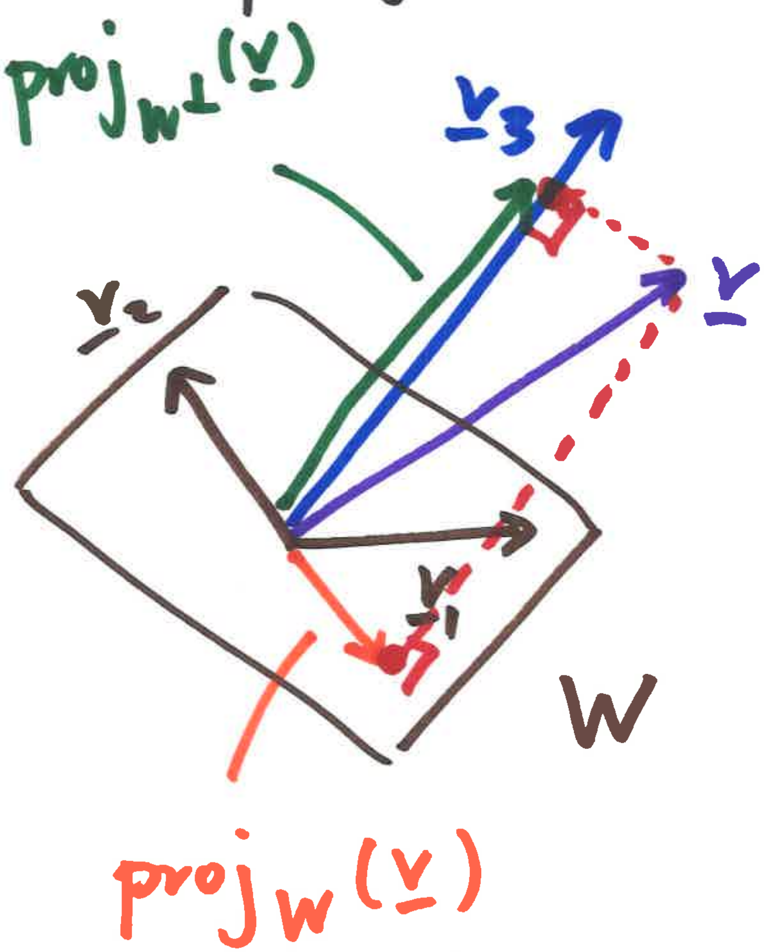
Conclusion: $(\underline{w}-\underline{w}') = \underline{0} = (\underline{z}-\underline{z}')$

so $\underline{w} = \underline{w}'$ $\underline{z} = \underline{z}'$

<u>Exer</u> Let $W = \text{span}\left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \subset \mathbb{R}^3$

Calculate $\text{proj}_W(\underline{v})$ where $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(<u>cartoon</u>

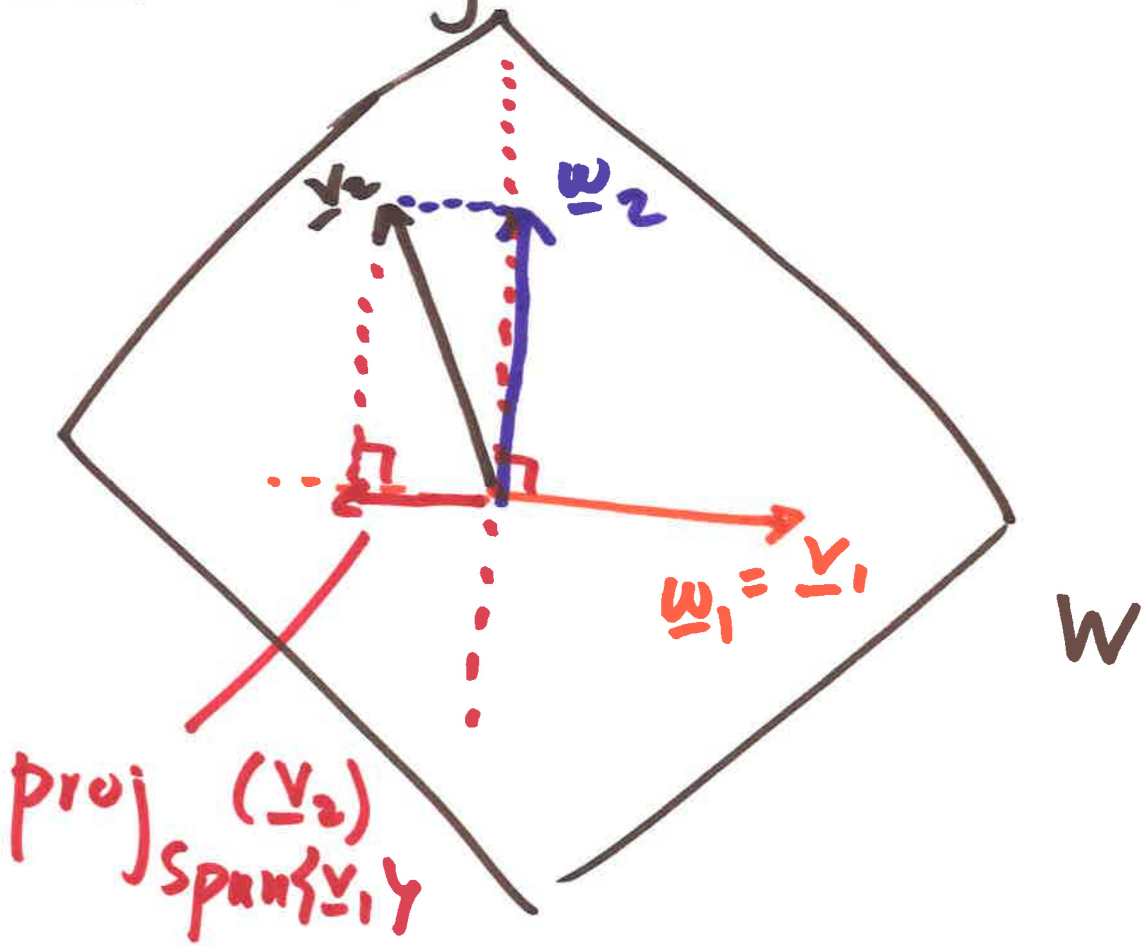


$\underline{v} = \text{proj}_W(\underline{v}) + \text{proj}_{W^\perp}(\underline{v})$

"any two determine the third"

Soln 1) Direct calculation:

Need orthog. basis of W!



$\underline{w}_1 = \underline{v}_1$

$\underline{w}_2 = \underline{v}_2 - \text{proj}_{\text{Span}\{\underline{v}_1\}}(\underline{v}_2)$

$$\underline{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{\underline{v}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{(-1)}{(2)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

This is our orthog basis!

Now $\text{proj}_W(\underline{v}) = \dfrac{\underline{v}\cdot\underline{w}_1}{\underline{w}_1\cdot\underline{w}_1}\underline{w}_1 + \dfrac{\underline{v}\cdot\underline{w}_2}{\underline{w}_2\cdot\underline{w}_2}\underline{w}_2$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{\frac{1}{2}}{\frac{3}{2}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{6} \\ -\frac{1}{2} + \frac{1}{6} \\ 0 - \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

2) <u>Indirect soln</u> using <u>Thm</u>

$$\text{proj}_W(\underline{v}) = \underline{v} - \text{proj}_{W^\perp}(\underline{v})$$

To calculate $\text{proj}_{W^\perp}(\underline{v})$, need orthog basis for $W^\perp$. But $W^\perp$ is a line so only need a vector

$$\underline{v}_3 \neq 0 \in W^\perp$$

For example take $\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\operatorname{proj}_W(\underline{v}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\underline{v} \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$