<u>Lecture 14</u> Diagonalization!

Or why we <u>love</u> e-vectors and e-values

<u>Friday</u> Quiz through § 5.2

"One person's craziness is another person's reality."

Tim Burton

Useful notation A $n \times n$ matrix
λ number

$$E_\lambda = \mathrm{Null}(A - \lambda I)$$ e-space for λ

Recall: λ is e-value of A

$$\iff$$

$$\dim E_\lambda > 0$$

<u>Exer</u> Suppose A is 3×3 matrix with

$$\chi_A(t) = (2-t)^3$$

For any number λ, what are possible dims of E_λ?

<u>Soln:</u> $E_\lambda \subset \mathbb{R}^3$ subspace so $\dim E_\lambda \leq 3$

Only e-value is $\lambda = 2$ so

$\dim E_\lambda = 0$ if $\lambda \neq 2$

$\dim E_\lambda > 0$ if $\lambda = 2$

Let's show by example that $\dim E_2$ could be 1, 2, or 3

$\underline{\dim E_2 = 3}$ $\quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (in fact only possibility)

$\underline{\dim E_2 = 2}$ $\quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (not unique possibility)

Basis <u>for</u> E_2 $\quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\dim E_2 = 1}$ $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (not unique possibility)

$\underline{\text{Basis for } E_2}$ $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

__Fact__: A $n \times n$ matrix with e-value λ

$$\boxed{1} \leq \dim E_\lambda \leq \boxed{\begin{array}{c}\text{multiplicity} \\ \text{of } \lambda \text{ as a} \\ \text{root of } \chi_A(t)\end{array}}$$

Exer Suppose A is a 6×6 matrix with $\chi_A(t) = (2-t)^3(1-t)t^2$. What are possible dims of e-spaces?

Soln E-values: $\lambda = 2, 1, 0$

mults: $3, 1, 2$

Possible dims of e-spaces: $1 \le \dim E_2 \le 3$, $1 \le \dim E_1 \le 1$, $1 \le \dim E_0 \le 2$

Now use e-values and e-vectors to find simpler representations of lin. transfs.

Favorite kinds of matrices:

1) **Diagonal matrices** $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

Note e-values of D $\lambda_1, \ldots, \lambda_n$

e-vectors $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

2) Diagonalizable matrices

An $n \times n$ matrix A is diagonalizable if there is a basis $\mathcal{B}$: $\underline{v}_1, \dots, \underline{v}_n$ of $\mathbb{R}^n$ such that the matrix of A with respect to $\mathcal{B}$ is diagonal.

$$[A]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [A\underline{v}_1]_{\mathcal{B}} & \cdots & [A\underline{v}_n]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

In other words, if we form change of coord matrix $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

then $$D = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \, A \, \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$$

is diagonal

Equivalently

$$A = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} \, D \, \underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$$

Thm A $n \times n$ matrix is diagonalizable

$$\iff$$

there is a basis $\underline{v}_1, \dots, \underline{v}_n$ of $\mathbb{R}^n$ consisting of e-vectors

Proof ($\Longrightarrow$) $D = {}_{B \leftarrow \mathcal{E}}P \; A \; {}_{\mathcal{E} \leftarrow B}P$

Take basis to be cols of ${}_{\mathcal{E} \leftarrow B}P$

why are cols $\underline{v}_i$ e-vectors?

Calculate

$$A\underline{v}_i = A\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\,\underline{e}_i\right) = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\,D\,\underline{e}_i =$$

Used: 1) $D = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}\,A\,\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$

2) $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}^{-1}$

$$= \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\,\lambda_i\underline{e}_i$$

$$= \lambda_i\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\,\underline{e}_i\right)$$

$$= \lambda_i\underline{v}_i$$

Yippee!

($\Leftarrow$) Set $\underset{\mathcal{E}\leftarrow B}{P} := \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

<u>Exercise</u>: check $\underset{B\leftarrow\mathcal{E}}{P} \, A \, \underset{\mathcal{E}\leftarrow B}{P}$ is diagonal.

<u>Exer</u> Is $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ diagonalizable?

<u>Soln</u> Need to find basis of e-vectors.

First find <u>e-values</u>: $\chi_A(t) = (1-t)(2-t) - 30$

$$= (7-t)(4+t)$$

e-<u>values</u> $\lambda = 7, -4$

<u>Note</u>: dim of each e-space must be 1.

Find e-spaces:

$\underline{\lambda = 7}$ $\quad E_7 = \text{Null}\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}$

$\underline{\text{Basis}}$: $\underline{v}_1 = \begin{bmatrix} +1 \\ 1 \end{bmatrix}$

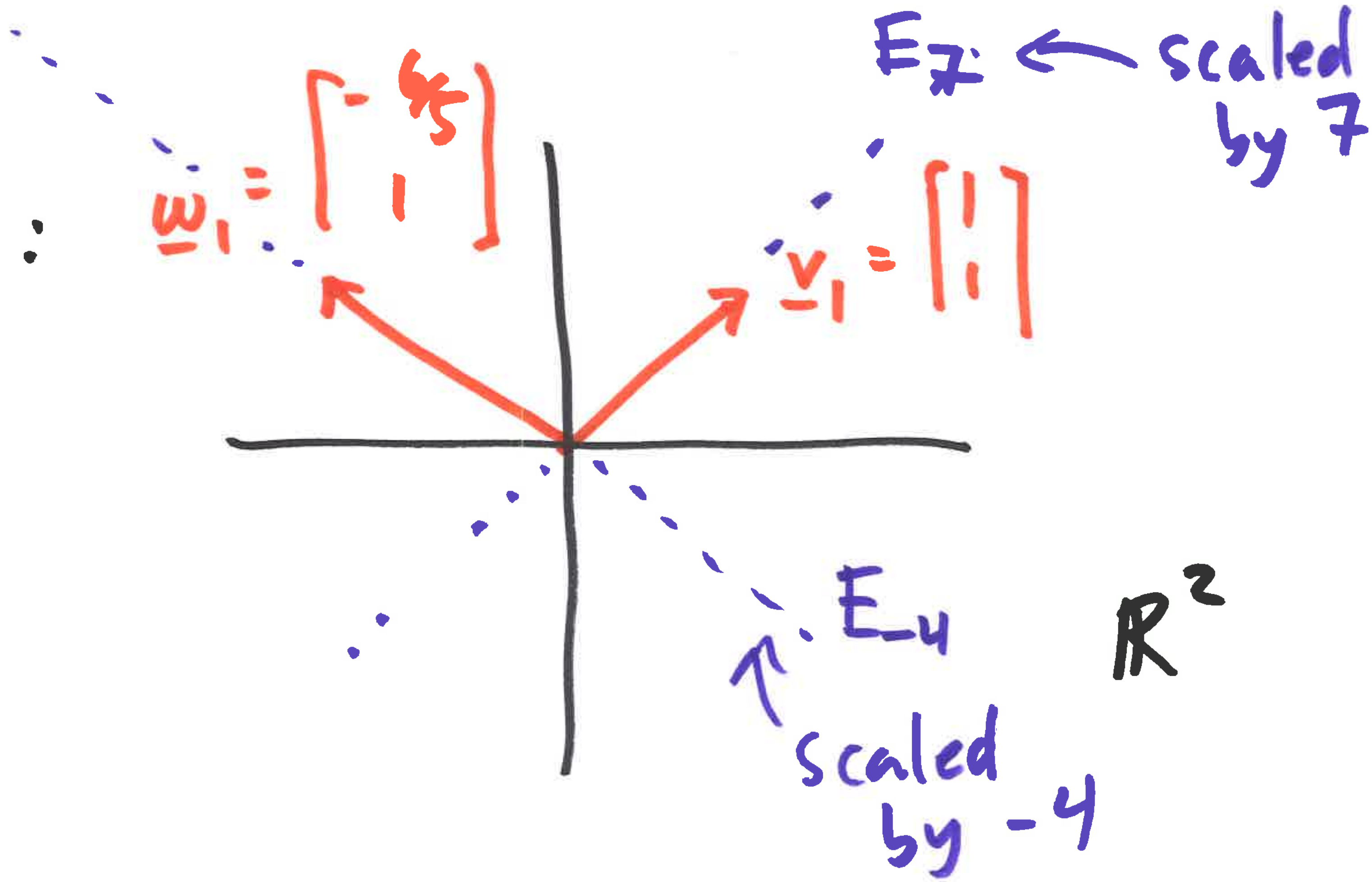
$\underline{\lambda = -4}$ $\quad E_4 = \text{Null}\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$

$\underline{\text{Basis}}$: $\underline{w}_1 = \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}$

Conclusion basis $\mathcal{B} = \{\underline{v}_1, \underline{w}_1\}$ of e-vectors

Then $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}\, A \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$

Cartoon:



Exer Is $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ diagonalizable?

Soln Find e-values: $\chi_A(t) = (2-t)(+1-t)^2$

$$\lambda = 2, -1$$

Possible dims of E_λ

$\lambda = 2$ $\quad \dim E_2 = 1$

$\lambda = +1$ $\quad \dim E_{+1} = 1$ or 2

Critical quantity:

1) Good news would be $\dim E_{+1} = 2$

enough e-vectors!

2) Bad news would be $\dim E_{+1} = 1$

not enough e-vectors...

won't be able to

diagonalize.

Need to calculate $E_1 = \text{Null}(A - 1 \cdot I)$

$$(A - 1 \cdot I) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

$\rightsquigarrow$ REF: 2 pivots

so E_1 has dim $= 1$

A not diagonalizable