Lecture 13 Eigenvectors and Eigenvalues!

Or How to Make Billions with Lin. Alg.

Today: Office Hours 1-3pm 736 Evans

Tues/Thurs: Discussion Group 2-3pm
115 Chavez

Friday: Quiz through §5.2

We are world's experts at solving linear systems:

$$\underbrace{A}_{m \times n \text{ matrix}} \; \underbrace{\underline{x}}_{n\text{-vector of vars}} = \underbrace{\underline{b}}_{m\text{-vector of eqn values}}$$

<u>New</u> eq<u>uation</u>: eigen<u>vector</u>/eigen<u>value</u>

number

$$A\,\underline{x} = \lambda\,\underline{x}$$

$n \times n$ matrix (square!)

n-vector of vars

If $A\underline{x} = \lambda \underline{x}$, with $\underline{x} \neq \underline{0}$,

then we say:

λ is an <u>eigenvalue</u> (e-value) of A

$\underline{x}$ is an <u>eigenvector</u> (e-vector) of A

Note: $\underline{x} = \underline{0}$ always solves the equation for any λ.

Groundrule: Never ever ever consider $\underline{x} = \underline{0}$ as a solution.

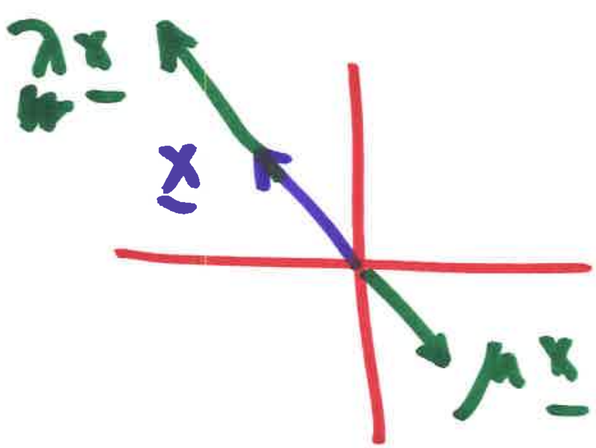
Note: 1) If $A\underline{x} = \lambda \underline{x}$ with $\underline{x} \neq \underline{0}$ and $A\underline{x} = \mu \underline{x}$ as well

then $\lambda = \mu$ since

$\lambda \underline{x} = A\underline{x} = \mu \underline{x}$ and

$\underline{x} \neq \underline{0}$ implies $\lambda = \mu$.

Cartoon



If $\lambda \neq \mu$ then $\lambda \underline{x} \neq \mu \underline{x}$

Conclusion If $\underline{x}$ is an e-vector then it has unique e-value

2) If $A\underline{x} = \lambda \underline{x}$ for some λ, then $\underline{x}$ is not unique.

can always scale by $c \neq 0$

Example: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Any $\underline{x} \in \mathbb{R}^2$ satisfies $A\underline{x} = 2\underline{x}$

So 2 is an e-value, and any $\underline{x} \neq \underline{0}$ is e-vector with e-value 2

Exer: Find e-values & e-vectors for

$$A = \begin{bmatrix} 7 & 0 & \cdots & & 0 \\ 0 & -1 & & & \vdots \\ \vdots & & -1 & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & & 0 & 2 \end{bmatrix}$$

(diagonal matrix)

Soln $\underline{\lambda = 7}$, $\underline{\lambda = 0}$, $\underline{\lambda = -1}$, $\underline{\lambda = 2}$

$$\underline{x} = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \\ 0 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 0 \\ a \\ b \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \end{bmatrix}$$

$a \neq 0$, $a \neq 0$, either $a \neq 0$ or $b \neq 0$, $a \neq 0$

General Strategy for solving $A\underline{x} = \lambda \underline{x}$

Special case: $\underline{\lambda = 0}$ $\quad A\underline{x} = \underline{0}$

We already know how to solve!

General case: any λ $\quad A\underline{x} = \lambda \underline{x}$

Equivalently solve $\quad A\underline{x} - \lambda \underline{x} = \underline{0}$

Rewrite as matrix eqn $\quad (A - \lambda I)\underline{x} = \underline{0}$

$$\left(\lambda I = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} \right)$$

We already know how to solve!

Find <u>null</u> space of $A - \lambda I$

<u>Terminology</u>: If λ is an e-value of A then we call $\mathrm{Null}(A-\lambda I)$ the <u>eigenspace</u> (e-space) of λ.

$$\mathrm{Null}(A-\lambda I) = \left\{ \begin{array}{c} \text{e-vectors} \\ \text{for } \lambda \end{array} \right\} \cup \{\underline{0}\}$$

To get started, we need to find possible e-values λ.

Note that λ is an e-value if and only if $\mathrm{Null}(A-\lambda I) \neq \{\underline{0}\}$

<u>Beautiful criterion</u>: $\mathrm{Null}(A-\lambda I) \neq \{\underline{0}\}$ if and only if λ solves the eqn

$$\chi_A(t) = \det(A - t \cdot I) = 0.$$

Terminology: 1) $\chi_A(t) = \det(A - tI)$ is called characteristic polynomial

2) $\chi_A(t) = 0$ is called characteristic equation

Summary of strategy:

1) Find e-values: solve $\chi_A(t) = \det(A - tI) = 0$

2) Find e-vectors: find $\mathrm{Null}(A - \lambda I)$ for e-values λ

<u>Exer</u> Find e-values & e-spaces for

1) $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$

1) Find <u>e-values</u>: solve

$$\chi_A(t) = \det(A - tI) = 0$$

$$(A - tI) = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} - t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-t & 3 \\ 0 & -1-t \end{bmatrix}$$

$$\chi_A(t) = \det(A - tI) = (2-t)(-1-t) - 3 \cdot 0$$

E-values: $\lambda = 2, -1$ since these are roots of $\chi_A(t) = 0$

2) $\underline{\lambda = 2}$ Find $\mathrm{Null}(A - 2I)$.

$$A - 2I = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}$$

Basis for e-space: $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = -1}$ Find Null$(A - (-1)I)$

$$A - (-1)I = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

<u>Basis for e-space</u>: $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

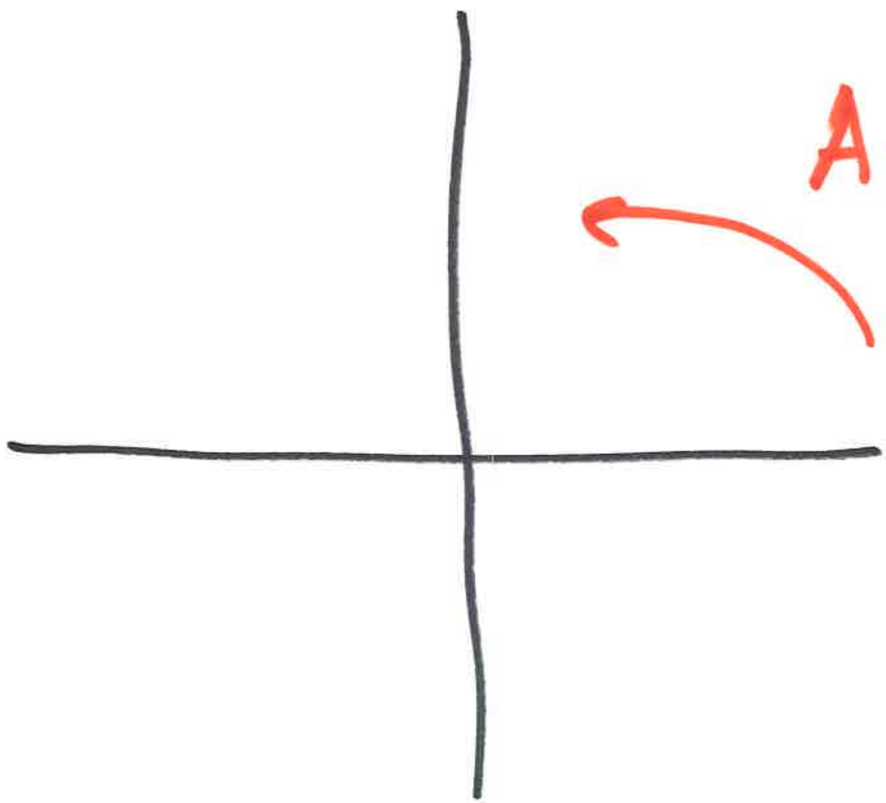
Find e-values: $\chi_A(t) = \det(A - tI)$

$$= \det \begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix} = t^2 + 1$$

No real solns to $\chi_A(t) = 0$!

No (real) e-values.

<u>Picture</u>: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



A rotates by $\frac{\pi}{2}$

No $\underline{x} \neq \underline{0}$ is taken to a scale of itself!

3) $A = \begin{bmatrix} 3 & 0 \\ -2 & 3 \end{bmatrix}$

<u>Find e-values</u>: $\chi_A(t) = \det(A - tI)$

$$= \det \begin{bmatrix} 3-t & 0 \\ -2 & 3-t \end{bmatrix} = (3-t)^2$$

$\underline{\lambda = 3}$ only e-value.

Find <u>e-space</u> Null$(A-3I)$

$$A-3I=\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$$

<u>Basis for e-space</u>: $\underline{v}_1=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$