Math 54 Final, Fall 2015

Name (Last, First): ______________________________

Student ID: ______________________________

GSI/Section: ______________________________

This is a closed book exam, no notes or calculators allowed. It consists of 9 problems, each worth 10 points. The lowest problem will be dropped, making the exam out of 80 points. Please avoid writing near the corner of the page where the exam is stapled, this area will be removed when the papers are scanned for grading. Put your extra work for each problem on the back of the page. If you run out of room on the back, use the pages attached at the back. If you want anything on those sheets graded, please indicate on the relevant problem which page your work is located on. **DO NOT REMOVE OR ADD ANY PAGES!**

1. (10 points)

(a) (2 points) State the rank theorem for an $m \times n$ matrix.

Solution: For an $m \times n$ matrix A, we have:

$$\text{Rank}(A) + \dim(\text{Null}(A)) = n$$

(b) (3 points) Use the rank theorem to show that a system of 8 homogenous linear equations in 10 variables must have a nontrivial solution.

Solution: A system of 8 linear equations in 10 variables corresponds to an 8×10 matrix, A. We are looking for nontrivial solutions to $Ax = 0$. That is, we want to show $\dim(\text{Null}(A)) > 0$. Since the rank is at most 8, and $\text{Rank}(A) + \dim(\text{Null}(A)) = 10$, we must have $\dim(\text{Null}(A)) \geq 2$.

(c) (5 points) Given an $n \times n$ matrix A, we have the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = Ax$. Use the rank theorem to prove that if T is injective (one-to-one) then it must be surjective (onto). Conversely, prove that if T is surjective then it must be injective. In both cases, conclude that T is an isomorphism.

Solution: If T is injective, that means $\dim(\text{Null}(A)) = 0$, which means $\text{Rank}(A) = n$, so T is also surjective. Conversely, $\text{Rank}(A) = n$, so $\dim(\text{Null}(A)) = 0$, so T is injective. In either case, T is injective and surjective, so T is an isomorphism.

2. (10 points) A 3×3 *Jordan block* is a matrix of the form $J_c = \begin{bmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{bmatrix}$, where c is some constant.

   (a) (3 points) Find all eigenvalues of J_c, and calculate the corresponding eigenspaces.

   Solution: The only eigenvalue of J_c is c, because J_c is triangular with c's on the diagonal. The eigenspace is the nullspace of the matrix $J_c - cI = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

   (b) (3 points) Prove that J_c is not diagonalizable.

   Solution: There is only one eigenvalue, with only a one-dimensional eigenspace. In order for a 3×3 matrix to be diagonalizable, the dimensions of the eigenspaces must add up to 3.

   (c) (4 points) Prove that the matrix $A_c = \begin{bmatrix} c & 1 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$ is neither diagonalizable nor similar to J_c.

   Solution: It is not diagonalizable because its only eigenspace is the nullspace of $A_c - cI = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is 2-dimensional and not 3-dimensional. To prove that it is not similar to J_c, recall that similar matrices have the same eigenvalues and the same dimensions of eigenspaces, but we have seen that the c-eigenspaces of J_c and A_c are respectively 1-dimensional and 2-dimensional.

3. (10 points) Mark the following as true or false. Justify your answers.

(a) (2 points) The Gram-Schmidt process can be used to turn a basis of eigenvectors into an orthogonal basis of eigenvectors.

Solution: False. Gram-Schmidt can turn a basis into an orthogonal basis, but it will not generally preserve the property that the basis is made of eigenvectors.

(b) (2 points) The matrix $\begin{bmatrix} 1 & 10 & 100 \\ 10 & 20 & 30 \\ 100 & 30 & 40 \end{bmatrix}$ has only real eigenvalues.

Solution: True. This matrix is symmetric.

(c) (2 points) If W is a subspace of an inner product space and $w_1, w_2, \dots w_n$ is a basis of it, then the projection to W can be written

$$\mathrm{proj}_W(v) = \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i. \tag{1}$$

Solution: False. This is only true if the w_i are an orthogonal basis.

(d) (2 points) The zero vector is orthogonal to every other vector.

Solution: True. $\langle 0, v \rangle = 0$ is one of the axioms of an inner product.

(e) (2 points) $\langle f, g \rangle = f(0)g(0) + f(1)g(1)$ is an inner product on the vector space of continuous real-valued functions on the interval $[0, 1]$.

Solution: False. One of the inner product axioms says that if $\langle f, f \rangle = 0$, then $f = 0$, and that isn't true here, for example if $f(x) = x^2 - x$.

4. (10 points) Let $M_{2\times 2}$ be the vector space of 2×2 matrices. Consider the linear transformation:

$$L : M_{2\times 2} \to M_{2\times 2},\ L(A) = A - A^T$$

(a) (3 points) Give a basis for $\mathrm{Ker}(L)$. Is L injective (one-to-one)?

Solution: The kernel is given by matrices A such that $A = A^T$, i.e. symmetric matrices. The space of symmetric matrices is 3 dimensional, spanned by the following three matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus L is not injective, since the kernel is nontrivial.

(b) (2 points) Give a basis for the image of L.

Solution: We know by the rank theorem that $\dim(Im(L)) = 1$, so it suffices to find a single nonzero vector in the image. We compute:

$$L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This vector is a basis for the image.

Let $\mathbb{P}_2$ and $\mathbb{P}_3$ denote the vector space of polynomials of degree less than or equal to 2 or 3 respectively. Consider the linear transformation:

$$S : \mathbb{P}_2 \to \mathbb{P}_3,\ S(f(x)) = (x+1) \cdot f(x)$$

(c) (2 points) Is S injective (one-to-one)? Justify.

Solution: Yes, S is injective, since if we had $(x+1)(f(x)) = 0$, then we must have $f(x) = 0$.

(d) (3 points) Is S surjective (onto)? Justify.

Solution: Since $\dim(\mathbb{P}_3) > \dim(\mathbb{P}_2)$, S is not surjective by the rank theorem.

5. (10 points) Find the general solution of the third-order differential equation $y''' - y'' + 4y' - 4y = e^t$.

Solution: First, we solve the associated homogeneous equation $y''' - y'' + 4y' - 4y = 0$. The auxiliary polynomial is $r^3 - r^2 + 4r - 4$, which factors as $(r-1)(r^2+4)$. So the roots are $r = 1$ and $r = \pm 2i$, and the homogeneous solutions are $y = c_1 e^t + c_2 \cos 2t + c_3 \sin 2t$.

To find a particular solution y_p, we would normally guess $y_p = Ae^t$, but we multiply this by t because $c_1 e^t$ is part of the homogeneous solution. This gives:

$$y_p = Ate^t \tag{2}$$
$$y_p' = A(te^t + e^t) \tag{3}$$
$$y_p'' = A(te^t + 2e^t) \tag{4}$$
$$y_p''' = A(te^t + 3e^t) \tag{5}$$
$$y_p''' - y_p'' + 4y_p' - 4y_p = 5Ae^t = e^t, \tag{6}$$

so $A = 1/5$, and a particular solution is $y_p(t) = \frac{te^t}{5}$.

Finally, the general solution is given by adding a the homogeneous and particular solutions: $y = \frac{te^t}{5} + c_1 e^t + c_2 \cos 2t + c_3 \sin 2t$.

6. (10 points) Let $y_1(t)$ and $y_2(t)$ be real-valued differentiable functions on $(-\infty, \infty)$.

(a) (2 points) Define what it means for $y_1(t)$ and $y_2(t)$ to be linearly independent on $(-\infty, \infty)$. Write your answer in complete sentences and be as precise as possible.

Solution: They are independent if whenever $c_1y_1(t) + c_2y_2(t) = 0$ for all t in $(-\infty, \infty)$, then $c_1 = c_2 = 0$.

(b) (2 points) Prove that t and e^t are linearly independent on $(-\infty, \infty)$.

Solution: Suppose $at + be^t = 0$ for all t. Then letting $t = 0$, we have $b = 0$. Letting $t = 1$ we get $a = 0$, (since $b = 0$). Thus $a = b = 0$ and the functions are independent.

For parts c) and d), answer true or false. If the statement is true, prove it. If the statement is false, provide a counterexample.

(c) (3 points) If $y_1(t)$ and $y_2(t)$ are linearly dependent, then $y_1'(t)$ and $y_2'(t)$ are linearly dependent.

Solution: True. If $c_1y_1(t) + c_2y_2(t) = 0$ for all t, with not both coefficients 0, then $c_1y_1'(t) + c_2y_2'(t) = 0$ for all t, where not both coefficients are 0.

(d) (3 points) If $y_1(t)$ and $y_2(t)$ are linearly independent, then $y_1'(t)$ and $y_2'(t)$ are linearly independent.

Solution: False. For example 1 and t are independent, but 0 and 1 are dependent.

7. (10 points) Consider $y'' + 2by' + y = 0$ where b is a real parameter. For which values of b does this differential equation have a nontrivial solution $y(t)$ such that:

$$\lim_{t\to\infty} y(t) = 0.$$

Solution: We look at the associated polynomial:

$$r^2 + 2br + 1 = 0$$
$$r = \frac{-2b \pm \sqrt{4b^2 - 4}}{2}$$

This has distinct real roots when $|b| > 1$, and repeated roots when $b = 1$, and complex roots when $|b| < 1$.

In the complex root case, the solutions have the form $y(t) = c_1 e^{-bt}\cos(\sqrt{1-b^2}t) + c_2 e^{-bt}\sin(\sqrt{1-b^2}t)$. These will only tend to 0 when $-b < 0$, so when $b > 0$, so when $\mathbf{0 < b < 1}$.

In the repeated root case, when $b = -1$ we have $y(t) = c_1 e^t + c_2 t e^t$, which doesn't tend to 0. When $\mathbf{b = 1}$, we have $c_1 e^{-t} + c_2 t e^{-t}$, which tends to 0 by L'Hopital's rule.

In the distinct real root case, we know that $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ where $\lambda_1 + \lambda_2 = -2b$, and $\lambda_1 \lambda_2 = 1$. The second equation tells us they have the same sign, and are thus both positive when b is negative and both negative when b is positive. Clearly the solution will only tend to 0 if they are both negative, so we see this happens when $\mathbf{b > 1}$.

In summation, by looking at the three cases, the solutions will tend to 0 when $\mathbf{b > 0}$.

8. (10 points) (a) (7 points) Let $f(x) = |x|$ for $-\pi < x < \pi$. Compute the Fourier series of $f(x)$ on $(-\pi, \pi)$.

Solution: Since $f(x)$ is even,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx dx = 0$$

On the other hand, $a_0 = \pi$. For $n \neq 0$, by applying integration by parts,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos nx dx = \frac{2}{\pi n^2}(\cos n\pi - 1) = \frac{2}{\pi n^2}((-1)^n - 1)$$

Then the Fourier series is

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n - 1)\cos nx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos((2n-1)x)$$

(b) (3 points) Use the previous part to compute the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Solution: Applying the point-wise convergence of the Fourier series at $x = 0$ gives:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

9. (10 points) This problem concerns solutions to the heat equation:

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = u(\pi,t) = 0$, and initial condition $u(x,0) = -5\sin(2x) + \sin(3x)$. Recall that $u(x,t)$ measures the temperature at a point x and a time t on a rod of length π.

(a) (4 points) Using separation of variables $u(x,t) = X(x)T(t)$ as usual, we end up having to solve

$$X''(x) + \lambda X(x) = 0$$

with boundary conditions $X(0) = 0, X(\pi) = 0$. For which $\lambda \geq 0$ does this boundary value problem have a nonzero solution? Give the corresponding fundamental solutions to the heat equation (without initial condition).

> **Solution:** As is carried out in the text, we have solutions when $\lambda = \frac{n^2\pi^2}{\pi^2} = n^2$. The solutions in this case end up being $\sin(nx)$ for $X(x)$, and e^{-5n^2t} for $T(t)$, giving fundamental solution $u_n(x,t) = \sin(nx)e^{-5n^2t}$.

(b) (2 points) Give the solution $u(x,t)$ to the heat equation above satisfying the initial condition $u(x,0) = f(x)$.

> **Solution:** Since $f(x)$ is already given as a sine series, we have:
>
> $$u(x,t) = -5u_2(x,t) + u_3(x,t) = -5sin(2x)e^{-20t} + sin(3x)e^{-45t}$$

(c) (2 points) For any *fixed* value of x in the interval $(0,\pi)$, and for $u(x,t)$ as in the previous part, compute the following:

$$\lim_{t\to\infty} u(x,t)$$

> **Solution:** For x fixed, we have:
>
> $$\lim_{t\to\infty} u(x,t) = 0$$
>
> since for x fixed, $u(x,t)$ is a linear combination of e^{-20t} and e^{-45t} which rapidly decay to 0 as $t \to \infty$. The physical interpretation of this is that as time goes on, the temperature of the entire rod approaches 0.

(d) (2 points) Show $u(\frac{\pi}{2}, t)$ is increasing for $t > 0$.

> **Solution:** We have $u(\frac{\pi}{2}, t) = -e^{-45t}$, which is increasing for $t > 0$, since the derivative is $45e^{-45t}$, which is positive for all t.