Welcome to Math 54 Review Part II!

ODE, PDE, Fourier Series

Universal Framework for (linear) ODE, PDE

V = vect sp of fns

Linear transf $T: V \to V$ involving derivatives

Solve $T\underline{x} = \underline{b}$, for $\underline{x}, \underline{b} \in V$

Subject $\underline{x}$ to IVP, BVP

__Ex__ 1) $V = \{$ fns on $[a,b]$ or $\mathbb{R}$ $\}$

(^ real-valued; $y(t)$)

$$T = \left(\frac{d}{dt}\right)^n + a_{n-1}\left(\frac{d}{dt}\right)^{n-1} + \cdots + a_0$$

general __situation__: a_i fns

concrete __situation__ : a_i consts

__Linear nth order ODE__

2) $V = \{$ vector-valued fns on $[a,b]$ or $\mathbb{R}\}$

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

$$T = I\frac{d}{dt} - A$$

<u>homog version</u> $T\underline{y} = 0$ means

$$\left(I\frac{d}{dt} - A\right)\underline{y} = 0$$

$$\underline{y}' = A\underline{y}$$

<u>nonhom version</u> $T\underline{y} = \underline{f}$ means . . .

$$\underline{y}' = A\underline{y} + \underline{f}$$

<u>general situation</u> A $n \times n$ - matrix of fns

<u>concrete situation</u> A $n \times n$ - matrix of consts

Linear <u>system</u> of ^ 1st <u>order</u> <u>O</u>DE

3) $V = \{$ fns on $\underset{x}{[0,L]} \times \underset{t}{\mathbb{R}}$
(^ real-valued)

or $\underset{x}{[-L,L]} \times \underset{t}{\mathbb{R}}$

or other possibilities... $\}$

notation for such a fn: $u(x,t)$

T = lin transf involving $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, \ldots$

favorite examples:

a) **Heat Eqn**: $T = \frac{\partial}{\partial t} - \beta \frac{\partial^2}{\partial x^2}$

$Tu = 0$ means $\frac{\partial}{\partial t} u = \beta \frac{\partial^2 u}{\partial x^2}$

b) **Wave Eqn**: $T = \frac{\partial^2}{\partial t^2} + \alpha^2 \frac{\partial^2}{\partial x^2}$

$Tu = 0$ means $\frac{\partial^2}{\partial t^2} u = -\alpha^2 \frac{\partial^2}{\partial x^2} u$

Skipping review of 1) n-th order linear ODE

Highlight challenge in 2) 1st order lin systs of ODE

$y' = Ay$ homog version

Two methods 1) Find e-values λ, e-vectors $\underline{u}$ of A

form solns: $e^{\lambda t}\underline{u}$

2) cols of e^{At} form basis of solns

Method 2) always works but can be computationally difficult

<u>Highlight issue with 1) Possible</u>

A does <u>not</u> have a basis of e-vectors

(This can not happen if A is sym
$A = A^T$

since such A have <u>orthogonal</u> <u>bases</u> of e-vectors)

If A does not have basis of e-vectors what can you do?

1) Find an e-vector for each e-value then solve lin syst for more e-vectors

2) Explore simplifying method 2) by considering $A - \lambda I$ for an e-value λ.
(This will always work for 2x2 A)

3) Take Math 110

Any nth order linear ODE can be reformulated as a 1st order linear system of ODE

$$y_1 = y$$

$$y_2 = y'$$

$$\vdots$$

$$y_n = y^{(n-1)}$$

... derive matrix A from given nth order ODE

Exercise 10.5 #9 Solve nonhomog Heat Eqn

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-x} \qquad L = \pi, \ \beta = 1$$

IVP $u(x,0) = \sin(2x)$

BVP $u(0,t) = 0 = u(\pi, t)$

Soln 1) Guess soln to nonhomog problem.

$Tu = f$ satisfying BVP

$$T = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \quad f = e^{-x}$$

__First attempt__ $u = e^{-x} \rightsquigarrow Tu = -e^{-x}$

__Second attempt__ $u = -e^{-x} \rightsquigarrow Tu = +e^{-x}$

Not out __of woods__ yet : $u = -e^{-x}$

does __no__t satisfy BVP

__Third attempt__ $u = -e^{-x} + a + bx \rightsquigarrow Tu = e^{-x}$

use BVP to solve for a & b

Want $u(0,t) = 0 = u(\pi,t)$

x = 0 $\quad u = -e^{-0} + a + b\cdot 0 = -1 + a$

so $a = 1$

x = π $\quad u = -e^{-\pi} + 1 + b\pi$

so $b = \dfrac{e^{-\pi} - 1}{\pi}$

Soln to nonhomog BVP: $\quad u = -e^{-x} + 1 + \left(\dfrac{e^{-\pi} - 1}{\pi}\right)x$

Now to solve IVP add in general soln to homog eqn

$$u = -e^{-x} + 1 + \left(\frac{e^{-\pi}-1}{\pi}\right)x + \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t}$$

(*)

$L = \pi$, $\beta = 1$

solve for c_n using Fourier formulas

__Want__

$$\sin(2x) = u(x,0) = -e^{-x} + 1 + \left(\frac{e^{-\pi}-1}{\pi}\right)x + \sum_{n=1}^{\infty} c_n \sin(nx)$$

$$\sin(2x) + e^{-x} - 1 - \left(\frac{e^{-\pi}-1}{\pi}\right)x = \sum_{n=1}^{\infty} c_n \sin(nx)$$

F sine series on $[0,\pi]$

Finally, calculate Fourier coeffs for

1) $\sin(2x) \rightsquigarrow c_{n,1}$

2) $e^{-x} \rightsquigarrow c_{n,2}$

3) $1 \rightsquigarrow c_{n,3}$

4) $x \rightsquigarrow c_{n,4}$

$$c_n = c_{n,1} + c_{n,2} - c_{n,3} - \left(\frac{e^{-\pi}-1}{\pi}\right) \cdot c_{n,4}$$

Then recombine to find total Fourier coeffs.

Put c_n's back into (*).