Solve \( T x = \beta \) for \( x, \beta \in V \) involving derivatives.

Linear transform \( L : V \to V \) with linear subspace of \( V \) = red sp of this.

Universal framework for ODE, PDE, ODE, PDE, Fourier Series.

Part 1

Welcome to Math 54 Review
Linear 1st Order ODE

Concrete situation: $a_i$ consts

General situation: $a_i$ fun

\[ y(t) = \left( \frac{d}{dt} \right)^n a_{n-1}(t) + \cdots + a_0 \]

Real-valued $y(t)$

\[ \forall x \in (a, b) \wedge \forall \phi \in \mathbb{R} \]

Subject to $x_0$ IVP, BVP
\[ y' = Ay + f \]

Homoq version \( Tg = 0 \) means

\( Tg = 0 \)

\( (Tg - A)g = 0 \)

\( T = I_A - A \)

\[ \begin{bmatrix} y_{m+1}(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} \]

2) \( V = \{ \text{vector-valued fun on } \mathbb{R}' \} \)
Linear system of ODE

1st order

System of constants

Concrete solution

General solution
\[ T = \text{LinTransf involving } e^{\frac{\pi}{\ell}}, \frac{\ell}{e} \]

\[ \text{for such a function: } 2 \pi (x' t) \]

or other possibilities...

\[ \mathbb{R} \times [0, \ell] \times [0, \ell] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \]

3) \( \forall \) \( f \) on real-valued \( [0, \ell] \)
\[ \frac{n^2 e^x}{e^2} = n^2 \frac{te}{e} \]

\[ \frac{x e}{e^2} + x = \frac{te}{e} = T \]

1) Wave Equation: \[ T_n = 0 \]

2) Heat Equation: \[ T_n = 0 \]

Arbitrary Examples:
Two methods
A, find eigenvalues and vectors

- 2) cols of \( \mathbf{A} \) form basis of solutions
- \( \mathbf{A} \) of \( \mathbf{u} \)
- Two methods
  1) Find eigenvalues, \( \mathbf{e} \), and vectors
  2) cols of \( \mathbf{A} \) form basis of solutions

\[ y' = A\mathbf{y} \] (Homo. Version)

Highly elliptical challenge in (2) 1st order linear ODE

Skipping review of (1) 1st order linear ODE
since such a have orthogonal bases of e-vectors,

This can not happen if A is symmetric

A does not have a basis of e-vectors

Highlight issue with Method 2: always works but can be computationally difficult
If A does not have basis of e-vectors, what can you do?

1) Find an e-vector for each e-value.
2) Then solve lin syst for more e-vectors.
   - by considering A - \lambda I
   - Explore simplifying method 2.

(This will always work for 2x2 A)

3) Take Math 110
Any $n^{th}$ order linear ODE can be reformulated as a 1st order linear system of ODEs. From a given matrix $A$ derive matrix $H$. ... 

\[ H_n = H^{(n-1)} \]
\[ H_2 = H' \]
\[ H_1 = H'' \]
\[ T = \frac{\alpha t - e^{-x}}{e} \quad f = e^{-x} \]

Solve (1) Guess solution to nonhomogeneous problem.

**Boundary Value Problem (BVP)**

\[ u(x, t) = 0 = u(1, t) \]

**Initial Value Problem (IVP)**

\[ u(x, 0) = \sin(2x) \]

\[ x = e^{2} \quad \frac{x e^{x}}{n e^{2}} = \frac{te}{ne} \]

Exercise 10.5. Solve homogeneous nonhomogeneous heat equation.
for a \& b

\text{use BVP to solve}

\text{Third attempt: } n = -e - x - x \rightarrow T_n = e - x

\text{First attempt: } n = e - x \rightarrow T_n = e - x

\text{Second attempt: } n = -e - x \rightarrow T_n = e - x

\text{Does not satisfy BVP}

\text{Not out of woods yet: } n = -e - x
\[
x = \frac{w}{1-1/n} + 1 - e\frac{-e}{1-1/n} - x
\]

\[
BVP
\]

Solve to nonhomogeneous

\[
\frac{n}{1-1/n} = 9
\]

So

\[
x = \frac{n}{1-1/n} + 1 - e
\]

So

\[
a = 1
\]

\[
b + a + 0 = -1 + 9
\]

\[
x = 0
\]

\[
\text{What } x(0, t) = 0 = u(t, 0)
\]
Using Fourier formulas

Solve for $c_n$

$1 = \int_0^L u(x) \, dx = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left( 1 - \frac{n^2 \pi^2}{x^2} \right) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2} \sin(n \pi x) e^{-n^2 \pi^2 t}$

$x = \frac{n \pi}{L}$

$\frac{x}{1-e^{-n^2 \pi^2 t}} + 1 + 1 - x$
\[
\int_{-\infty}^{\infty} \left( \frac{\pi}{x^2 + 1} \right) + 1 - e^{-x} = \xi_1 (x) \quad (x \to 0) \quad \text{Want} \quad n = \xi_1 (x)
\]
Put $c_n$ back into $(x)$.

Then regrouping to find total Fourier:

$$c_n = (\frac{n!}{e^{\pi n/2}} - c_n, 3) - (c_n, 3) + (c_n, 1)$$

$$c_n, 4 \rightarrow c_n, 2 \rightarrow c_n, 2$$

$$c_n, 1 \rightarrow c_n, 1$$

Finally, calculate Fourier coefficients for.