Name (Last, First): ______________________________

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1. Consider the matrix
$$A = \begin{pmatrix} 5 & 5 \\ -13 & -3 \end{pmatrix}.$$
Use a change of basis to represent A as a rotation and scaling transformation. In other words, find a real matrix
$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
and an invertible real matrix P such that $A = PCP^{-1}$.

Solution. The characteristic equation is

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & 5 \\ -13 & -3-\lambda \end{pmatrix}$$

$$= (5-\lambda)(-3-\lambda) + 65 = \lambda^2 + 2\lambda + 50.$$

This has two complex zeros
$$\lambda_\pm = 1 \pm \sqrt{1-50} = 1 \pm 7i.$$
We can choose the eigenvalue $\lambda_- = 1 - 7i$ and find an associated eigenvector:

$$\text{Nul}(A - \lambda_- I) = \text{Nul} \begin{pmatrix} 4+7i & 5 \\ -13 & -4+7i \end{pmatrix} = \text{Nul} \begin{pmatrix} 4+7i & 5 \\ 0 & 0 \end{pmatrix}$$

(we didn't really row reduce here; the fact that λ_- was an eigenvalue tells us that there must be a row of zeros in the REF). We can choose the eigenvector $v = \begin{pmatrix} -5 \\ 4+7i \end{pmatrix}$, since

$$\begin{pmatrix} 4+7i & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 4+7i \end{pmatrix} = \begin{pmatrix} -5(4+7i) + 5(4+7i) \\ 0 \end{pmatrix} = 0.$$

Then Theorem 9 tells us that $A = PCP^{-1}$, where

$$P = \begin{pmatrix} \text{Re}[v] & \text{Im}[v] \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 4 & 7 \end{pmatrix}$$

and
$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 7 & 1 \end{pmatrix}, \text{ (where } a - ib = \lambda_- = 1 - 7i).$$

□

2. Inside of $\mathbb{R}^4$, consider the vectors

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find all vectors that are simultaneously orthogonal to v_1, v_2, and v_3 with respect to the dot product.

Solution. First, let us find a basis of the null space

$$\text{Nul} \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} = \text{Nul} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \text{Nul} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$= \text{Nul} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$= \text{Nul} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

A basis is given by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -2 \end{pmatrix}$, and all scalings of this vector are exactly those vectors that are simultaneously orthogonal to v_1, v_2 and v_3. □