Name (Last, First): ______________________________

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1. Is the set W of 2×2 symmetric matrices a subspace of the vector space V of all 2×2 matrices?

(Recall that a matrix A is symmetric if and only if $A^T = A$. Equivalently, a symmetric 2×2 matrix is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.)

There are two ways of solving this problem, choose your favorite one!

<u>Solution 1:</u> All we need to show that the zero-matrix is in W, and that W is closed under addition and scalar multiplication.

Zero-vector: The 2×2 zero-matrix $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $O^T = O$, therefore O is in W

Closed under addition: Suppose A and B are in W. Then $A^T = A$ and $B^T = B$. But then, by properties of transpose:

$$(A+B)^T = A^T + B^T = A + B$$

And therefore $A + B$ is symmetric. Since A and B were arbitrary in W, it follows that W is closed under addition

Closed under scalar multiplication: Suppose A in in W and c is a real number, then $A^T = 0$, and so, by properties of transposes:

$$(cA)^T = c\left(A^T\right) = cA$$

And therefore cA is symmetric. Since A and c were arbitrary, it follows that W is closed under multiplication.

Therefore, W is a subspace of V.

<u>Solution 2:</u>

Notice that every matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ can be written as:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, it follows that:

$$W = Span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

But since the span of any number of vectors in V is a subspace of V, it follows that W is a subspace of V.

2. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$.

a. Calculate the change-of-coordinates matrix $P_{\mathcal{B}}$ from $\mathcal{B}$ to the standard basis of $\mathbb{R}^2$.

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}$$

b. Use part a. to calculate $[\mathbf{x}]_{\mathcal{B}}$ given $\mathbf{x} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$

We know that:

$$\mathbf{x} = P_{\mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}}$$

And therefore:

$$[\mathbf{x}]_{\mathcal{B}} = (P_{\mathcal{B}})^{-1} \, \mathbf{x} = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 15 \\ -10 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$