1. Determine if the columns of the matrix form a linearly independent set.

\[
\begin{bmatrix}
0 & 2 & 3 \\
1 & 3 & 6 \\
-1 & 1 & 0
\end{bmatrix}
\]

Solution 1. **The set is linearly dependent.** This is because

\[
3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

so that the three columns form a linearly dependent set.

Solution 2. Let’s take the associated augmented matrix for the homogeneous equation.

\[
\begin{bmatrix}
0 & 2 & 3 & 0 \\
1 & 3 & 6 & 0 \\
-1 & 1 & 0 & 0
\end{bmatrix}
\]

In order to make the matrix to be in Row Echelon Form, we need the below row reduction steps.

\[
\begin{bmatrix}
0 & 2 & 3 & 0 \\
1 & 3 & 6 & 0 \\
-1 & 1 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

The first \( \sim \) is obtained by interchanging 1st and 2nd rows.
The second \( \sim \) : Changing 3rd row into 1st row + 3rd row.
The last \( \sim \) : Changing 3rd row into 3rd row + \((-2)\)× 2nd row.

Hence, we can find a solution \( z \) =free from 3rd row. \( y = -\frac{3}{2}z \) from 2nd row. \( x = y \) from 1st and 2nd row. Find one example of \( x, \ y, \) and \( z \) by setting \( z = -2 \), we get the weights \( x, \ y, \) and \( z \) (not all zeros) that make the linear combination \( x\mathbf{x}_1 + y\mathbf{x}_2 + z\mathbf{x}_3 \) become zero.
2. Let $T(x, y) = (2x + y, x)$. Show that $T$ is a one-to-one linear transformation. Does $T$ map $\mathbb{R}^2$ onto $\mathbb{R}^2$?

Solution. First of all,

$$T((x_1, y_1)) + T((x_2, y_2)) = (2x_1 + y_1, x_1) + (2x_2 + y_2, x_2)$$
$$= (2(x_1 + x_2) + (y_1 + y_2), x_1 + x_2) = T((x_1 + x_2, y_1 + y_2))$$

Also, $T(c(x, y)) = (2cx + cy, cx) = cT((x, y))$. Hence, $T$ is a linear transformation.

Now, in order to prove that $T$ is one-to-one, (because $T$ is a linear transformation and by Theorem 11 (Chapter 1.9)) we only need to show that

If $T(x, y) = (0, 0)$ then $x = 0$ and $y = 0$.

Suppose that $T(x, y) = (0, 0)$, then it implies that $2x + y = 0$, $x = 0$. So, obviously, you get $x = 0$ and $y = 0$. Henceforth, $T$ is a one-to-one linear transformation.

For the last question, the answer is YES. To get this answer, we need the argument below.

For an arbitrary element in $\mathbb{R}^2$, say $(z, w)$, if we define $x = w$, $y = z - 2w$ then

$$T(x, y) = (2x + y, x) = (2w + z - 2w, w) = (z, w).$$

Therefore, $T$ maps $\mathbb{R}^2$ onto $\mathbb{R}^2$. 