Name (Last, First): ____________________

Student ID: ____________________

1. Find a general solution to the homogeneous equation:

$$\left(\frac{d}{dt} - 5\right)^3 \left(\frac{d^2}{dt^2} + 4\right) y = 0$$

Solution. The auxiliary equation becomes

$$(r-5)^3(r^2+4) = 0$$

with the roots

$$\begin{cases} r = 5 & \text{(triple)} \\ r = 2i & \text{(single)} \\ r = -2i & \text{(single)} \end{cases}$$

For the triple root $r = 5$, we obtain three linearly independent solutions

$$y_1 = e^{5t},\ y_2 = te^{5t},\ y_3 = t^2e^{5t}.$$

The complex roots $r = \pm 2i$ correspond to

$$y_4 = \cos 2t,\ y_5 = \sin 2t$$

Hence, a general solution y consists of all possible linear combinations of those five linearly independent solutions:

$$y = C_1e^{5t} + C_2te^{5t} + C_3t^2e^{5t} + C_4\cos 2t + C_5\sin 2t$$

where $C_1, \cdots, C_5$ are arbitrary.

2. Let

$$\mathbf{x}_1 = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Determine if $\{\mathbf{x}_1, \mathbf{x}_2\}$ form a fundamental solution set of the system:

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

Solution. First, show that $\mathbf{x}_1$ and $\mathbf{x}_1$ are solutions to the system by checking

$$\mathbf{x}_1' = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}' = \begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1$$

$$\mathbf{x}_2' = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}' = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_2$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to a homogeneous linear system, it suffices to check if the Wronskian is equal to zero at some point t_0 in order to determine the linear independence of $\{\mathbf{x}_1, \mathbf{x}_2\}$. Then the Wronskian reads

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix}$$

Choose any point t_0, say $t_0 = 0$, so that

$$W[\mathbf{x}_1, \mathbf{x}_2](0) = \det \begin{bmatrix} -\sin 0 & \cos 0 \\ \cos 0 & \sin 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$$

Therefore, the Wronskian is always nonzero at any t, which implies that the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Hence, $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a fundamental solution set of the system.