Name (Last, First): ______________________________

Student ID: ______________________________

1. Find an orthogonal matrix P and diagonal matrix D such that $A = PDP^{-1}$ where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Solution. The characteristic equation is $\det \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda - 5 = 0$

Therefore eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$

The eigenvector for $\lambda_1 = 5$ will be nonzero vectors in $Nul \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$, we can pick $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvector for $\lambda_2 = -1$ will be nonzero vectors in $Nul \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$, we can pick $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We know that for a symmetric matrix, eigenvectors with different eigenvalues are orthogonal to each other, and so to orthogonally diagonalize the matrix, we only have to normalize the eigenvectors to unit vectors.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix},$$

So $P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. □

2. Solve the initial value problem

$$y'' + 6y' + 5y = 0, y(0) = 1, y'(0) = 1$$

Solution. The auxiliary equation is $\lambda^2 + 6\lambda + 5 = 0$, the solutions are $-1, -5$, since we have two different real solutions, the general solution to the equation will be $C_1 e^{-x} + C_2 e^{-5x}$. Plug in the initial condition, we have:

$$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 - 5C_2 &= 1 \end{aligned}$$

Solve the equation above, we have $C_1 = \frac{3}{2}, C_2 = -\frac{1}{2}$, so the solution is $\frac{3}{2}e^{-x} - \frac{1}{2}e^{-5x}$

□