Name (Last, First): ____________________

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1. Find the set of all $\mathbf{x}$ in $\mathbb{R}^2$ minimizing $\|A\mathbf{x} - \mathbf{b}\|$ where

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

Solution. This is equivalent to solving the least-squares problem for $A\mathbf{x} = \mathbf{b}$, which we can do by solving the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. We can compute

$$A^T A = \begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 12 \\ 9 \end{bmatrix}.$$

Then $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ has the unique solution

$$\hat{\mathbf{x}} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix},$$

which is the unique value of $\mathbf{x}$ minimizing $\|A\mathbf{x} - \mathbf{b}\|$.

2. Let H be the subspace of $\mathbb{R}^4$ given by

$$H = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Find an orthonormal basis for H.

Solution. Let v_1, v_2, v_3 be the three vectors given above that span H. We'll use the Gram-Schmidt process:

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}.$$

Then $\{u_1, u_2, u_3\}$ is an orthogonal basis for H. To produce an orthonormal basis, we normalize these vectors, yielding the set

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\}$$