Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points. We will grade all 6 problems, and count your top 5 scores.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total Possible	50	

Problem 1) Decide if the following statements are ALWAYS TRUE or SOMETIMES FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer boxes of the chart. (Correct answers receive 2 points, incorrect answers or blank answers receive 0 points.)

Statement	1	2	3	4	5
Answer	TRUE	FALSE	TRUE	TRUE	TRUE

1) If A is an $n \times n$ diagonalizable matrix with a single eigenvalue λ, then $A = \lambda I_n$.

2) If A and B are diagonalizable $n \times n$ matrices, then $A + B$ is also diagonalizable.

3) If the columns of an $n \times n$ matrix are orthonormal, then its rows are as well.

4) If A and B are similar $n \times n$ matrices, then they have the same rank.

5) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^n$ with $\mathbf{v}_1$ orthogonal to $\mathbf{v}_2$. Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal set if and only if the projection of $\mathbf{v}_3$ to the span of $\mathbf{v}_1, \mathbf{v}_2$ is the zero vector.

Problem 2) Indicate with an **X** in the chart all of the answers that satisfy the questions below. You do not need to justify your answers. It is possible that any number of the answers satisfy the questions. (A completely correct row of the chart receives 2 points, a partially correct row receives 1 point, but any incorrect X in a row leads to 0 points.)

	(a)	(b)	(c)	(d)	(e)
Question 1	X	X	X		
Question 2	X			X	X
Question 3	X	X			
Question 4		X		X	
Question 5	X			X	X

1) Let A be an $n \times n$ matrix. Which of the following is equivalent to the statement: A is not invertible?

$a)$ 0 is an eigenvalue of A.
$b)$ 1 is an eigenvalue of $I_n - A$.
$c)$ 0 is a solution of $\det(A - \lambda I_n) = 0$.
$d)$ $\det(A - \lambda I_n) = (1 - \lambda)^n$.
$e)$ A is diagonalizable.

2) Which of the following matrices are diagonalizable with real eigenvalues?

$$a) \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad b) \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \quad c) \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad d) \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad e) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

3) Which of the following linear transformations is an isomorphism?

$$a)\ P_2 \to \mathbb{R}^3 \quad p(x) \mapsto \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix}$$

$$b)\ P_2 \to \mathbb{R}^3 \quad p(x) \mapsto \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$

$$c)\ \mathbb{R}^3 \to P_2 \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto p(x) = (a-b) + (b-c)x + (c-a)x^2$$

$$d)\ \mathbb{R}^3 \to P_2 \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto p(x) = (a+b+c) + (b+c)x + ax^2$$

$$e)\ P_2 \to P_2 \quad T(p(x)) = p'(x) + x^2 p''(x)$$

4) Which of the following lists of vectors in $\mathbb{R}^3$ is an orthonormal set?

$$a)\ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad b)\ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \quad c)\ \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$d)\ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \quad e)\ \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

5) Suppose A is a 4×3 matrix whose columns are orthogonal. Which of the following matrices must also have orthogonal columns?

a) The reduced row echelon form of A.
b) A^T.
c) AA^T.
d) $A^T A$.
e) $-A$.

Problem 3) Consider the 4×4 matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ -3/2 & 3/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (8 points) Find 4×4 matrices P and D, with P invertible and D diagonal, such that

$$A = PDP^{-1}$$

Solution: Characteristic equation: $\lambda^4 - \lambda^3 - 2\lambda^2 = \lambda^2(\lambda^2 - \lambda - 2) = \lambda^2(2-\lambda)(-1-\lambda) = 0$.

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

b) (2 points) Find a vector $\mathbf{x}$ such that $A^{54}\mathbf{x} = \mathbf{x}$.

Solution: Take $\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ the eigenvector for $\lambda = -1$.

Problem 4) Recall that given a basis B of a vector space V, a basis C of a vector space W, and a linear transformation $T : V \to W$, we can assign a matrix $[T]$ such that

$$[T\mathbf{x}]_C = [T][\mathbf{x}]_B, \qquad \text{for all } \mathbf{x} \text{ in } V.$$

a) (6 points) Find a basis B of $V = P_1$ and a basis C of $W = P_2$ such that

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is the matrix of the linear transformation

$$T : V \to W \qquad T(p(x)) = \frac{d}{dx}p(x) - xp(x)$$

Solution: Take $B = \{1, x\}$, $C = \{-x, 1 - x^2, 1\}$.

b) (4 points) Is it possible to find bases such that

$$[T] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}?$$

Either find such bases or justify why it is not possible.

Solution: It is not possible: the rank of the original transformation is 2 so any matrix representing it must have rank 2 as well.

Problem 5) Consider the subspace W of $\mathbb{R}^3$ spanned by

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

a) (5 points) Find a nonzero vector $\mathbf{w}$ in W orthogonal to $\mathbf{u}$.

Solution: Set $\mathbf{w} = a\mathbf{u}+b\mathbf{v}$. We need $\mathbf{w}\cdot\mathbf{u} = 0$. We calculate $\mathbf{w}\cdot\mathbf{u} = a\mathbf{u}\cdot\mathbf{u}+b\mathbf{v}\cdot\mathbf{u} = 2a+(-1)b$. So we need $b = 2a$. Take $\mathbf{w} = \mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

b) (5 points) Find the vector in W closest to the vector

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Solution: We need $proj_W(\mathbf{y})$.

By part a) above, W is the span of the orthogonal set $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Thus

$$proj_W(\mathbf{y}) = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \frac{\mathbf{y}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{4}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{-2}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -7/3 \\ 2/3 \end{bmatrix}$$

Problem 6) Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. If always true, give a proof. If sometimes false, give a counterexample.

a) (5 points) Assertion: Suppose the characteristic equation of an 2×2 matrix is $\lambda^2 = 0$ and A^2 is diagonalizable. Then A is diagonalizable.

Solution: False. Counterexample:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

b) (5 points) Assertion: Suppose there are $n \times n$ matrices A, P and D, with P invertible and D diagonal, such that

$$A = PDP^{-1}$$

Then the columns of P are eigenvectors for A.

Solution: True. Let $\mathbf{v}_i$ be the ith column of A. (Since P is invertible, $\mathbf{v}_i$ is nonzero.) Note that $P\mathbf{e}_i = \mathbf{v}_i$ and $P^{-1}\mathbf{v}_i = \mathbf{e}_i$. Let λ_i be the ith diagonal entry of D so that $D\mathbf{e}_i = \lambda_i \mathbf{e}_i$.

Then we have

$$A\mathbf{v}_i = PDP^{-1}\mathbf{v}_i = PD\mathbf{e}_i = P\lambda_i\mathbf{e}_i = \lambda_i P\mathbf{e}_i = \lambda_i\mathbf{v}_i.$$