This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points. We will grade all 6 problems, and count your top 5 scores.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Maximum Score</th>
<th>Your Score</th>
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<td>Total Possible</td>
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**Problem 1**) Decide if the following statements are ALWAYS TRUE or SOMETIMES FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer boxes of the chart. (Correct answers receive 2 points, incorrect answers or blank answers receive 0 points.)

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<tr>
<th>Statement</th>
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<tr>
<td>Answer</td>
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1) If $A$ is an $n \times n$ diagonalizable matrix with a single eigenvalue $\lambda$, then $A = \lambda I_n$.  

2) If $A$ and $B$ are diagonalizable $n \times n$ matrices, then $A + B$ is also diagonalizable.  

3) If the columns of an $n \times n$ matrix are orthonormal, then its rows are as well.  

4) If $A$ and $B$ are similar $n \times n$ matrices, then they have the same rank.  

5) Suppose $v_1, v_2, v_3$ are vectors in $\mathbb{R}^n$ with $v_1$ orthogonal to $v_2$. Then $v_1, v_2, v_3$ form an orthogonal set if and only if the projection of $v_3$ to the span of $v_1, v_2$ is the zero vector.
**Problem 2** Indicate with an X in the chart all of the answers that satisfy the questions below. You do not need to justify your answers. It is possible that any number of the answers satisfy the questions. (A completely correct row of the chart receives 2 points, a partially correct row receives 1 point, but any incorrect X in a row leads to 0 points.)

<table>
<thead>
<tr>
<th>Question</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
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<tbody>
<tr>
<td>Question 1</td>
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<td>Question 2</td>
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1) Let $A$ be an $n \times n$ matrix. Which of the following is equivalent to the statement: $A$ is not invertible?

   a) 0 is an eigenvalue of $A$.
   b) 1 is an eigenvalue of $I_n - A$.
   c) 0 is a solution of $\det(A - \lambda I_n) = 0$.
   d) $\det(A - \lambda I_n) = (1 - \lambda)^n$.
   e) $A$ is diagonalizable.

2) Which of the following matrices are diagonalizable with real eigenvalues?

   a) \[
   \begin{bmatrix}
   1 & 3 \\
   3 & 1 \\
   \end{bmatrix}
   \]
   b) \[
   \begin{bmatrix}
   1 & -3 \\
   3 & 1 \\
   \end{bmatrix}
   \]
   c) \[
   \begin{bmatrix}
   1 & 0 & 0 \\
   1 & 1 & 0 \\
   3 & 2 & 1 \\
   \end{bmatrix}
   \]
   d) \[
   \begin{bmatrix}
   -1 & 0 & 0 \\
   1 & 0 & 0 \\
   1 & 1 & 1 \\
   \end{bmatrix}
   \]
   d) \[
   \begin{bmatrix}
   0 & 0 & 1 \\
   0 & 1 & 0 \\
   1 & 0 & 0 \\
   \end{bmatrix}
   \]
3) Which of the following linear transformations is an isomorphism?

\[ a) \quad P_2 \to \mathbb{R}^3 \quad p(x) \mapsto \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix} \]

\[ b) \quad P_2 \to \mathbb{R}^3 \quad p(x) \mapsto \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix} \]

\[ c) \quad \mathbb{R}^3 \to P_2 \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto p(x) = (a - b) + (b - c)x + (c - a)x^2 \]

\[ d) \quad \mathbb{R}^3 \to P_2 \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto p(x) = (a + b + c) + (b + c)x + ax^2 \]

\[ e) \quad P_2 \to P_2 \quad T(p(x)) = p'(x) + x^2p''(x) \]

4) Which of the following lists of vectors in \( \mathbb{R}^3 \) is an orthonormal set?

\[ a) \quad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \]

\[ b) \quad \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \]

\[ c) \quad \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \]

\[ d) \quad \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \]

\[ e) \quad \begin{bmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix} \]

5) Suppose \( A \) is a \( 4 \times 3 \) matrix whose columns are orthogonal. Which of the following matrices must also have orthogonal columns?

\[ a) \quad \text{The reduced row echelon form of } A. \]

\[ b) \quad A^T. \]

\[ c) \quad AA^T. \]

\[ d) \quad A^T A. \]

\[ e) \quad -A. \]
Problem 3) Consider the $4 \times 4$ matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ -3/2 & 3/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (8 points) Find $4 \times 4$ matrices $P$ and $D$, with $P$ invertible and $D$ diagonal, such that

$$A = PDP^{-1}$$

b) (2 points) Find a vector $x$ such that $A^{54}x = x$. 
Problem 4) Recall that given a basis $B$ of a vector space $V$, a basis $C$ of a vector space $W$, and a linear transformation $T : V \to W$, we can assign a matrix $[T]$ such that

$$[Tx]_C = [T][x]_B, \quad \text{for all } x \text{ in } V.$$ 

a) (6 points) Find a basis $B$ of $V = P_1$ and a basis $C$ of $W = P_2$ such that

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is the matrix of the linear transformation

$$T : V \to W \quad T(p(x)) = \frac{d}{dx}p(x) - xp(x)$$

b) (4 points) Is it possible to find bases such that

$$[T] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}?$$

Either find such bases or justify why it is not possible.
Problem 5) Consider the subspace $W$ of $\mathbb{R}^3$ spanned by

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

a) (5 points) Find a nonzero vector $w$ in $W$ orthogonal to $u$.

b) (5 points) Find the vector in $W$ closest to the vector

$$y = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$
Problem 6) Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. If always true, give a proof. If sometimes false, give a counterexample.

a) (5 points) Assertion: Suppose the characteristic equation of an $2 \times 2$ matrix is $\lambda^2 = 0$ and $A^2$ is diagonalizable. Then $A$ is diagonalizable.

b) (5 points) Assertion: Suppose there are $n \times n$ matrices $A$, $P$ and $D$, with $P$ invertible and $D$ diagonal, such that

$$A = PDP^{-1}$$

Then the columns of $P$ are eigenvectors for $A$. 
