This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points, of which you must complete 5. **Choose one problem not to be graded by crossing it out in the box below.** If you forget to cross out a problem, we will roll a die to choose one for you.

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1) Decide if the following statements are ALWAYS TRUE (T) or SOMETIMES FALSE (F). You do not need to justify your answers. (Correct answers receive 2 points, incorrect answers -2 points, blank answers 0 points.)

a) If $v_1, v_2, v_3, v_4$ are linearly independent vectors in $\mathbb{R}^6$, then $v_1 + v_2, v_3 - v_4$ are linearly independent vectors.

(T) If $a(v_1 + v_2) + b(v_3 - v_4) = 0$ then $av_1 + av_2 + bv_3 - bv_4 = 0$ so $a = b = 0$.

b) The following linear system is inconsistent

\[
\begin{align*}
-2x_1 + 4x_2 - 6x_3 + 8x_4 &= 10 \\
x_1 - 2x_2 + 3x_3 - 4x_4 &= -5
\end{align*}
\]

(F) The system can be written as an augmented matrix and put in reduced row echelon form

\[
\begin{bmatrix}
-2 & 4 & -6 & 8 & | & 10 \\
1 & -2 & 3 & -4 & | & -5
\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}
1 & -2 & 3 & -4 & | & -5 \\
0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]

There is no row of the form $[0 0 0 0 | 1]$.

c) If $A$ is a $3 \times 2$ matrix and $B$ is a $2 \times 3$ matrix, then the rank of the $3 \times 3$ matrix $AB$ must be less than or equal to 2.

(T) Since $B$ is $2 \times 3$, we have $\text{rank}(B) \leq 2$, and so $\dim \text{Nul}(B) \geq 1$. Hence $\dim \text{Nul}(AB) \geq 1$, and so $\text{rank}(AB) \leq 2$.

d) If two $m \times n$ matrices $A$ and $B$ have the same reduced row echelon form, then they have the same column spaces.

(F) Counterexample:

\[
A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

e)\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 3 & 1 \\
1 & 1 & 1 & 4 \\
1 & 1 & 1 & 5
\end{vmatrix} = 24
\]

(T) Row reduce to an upper triangular matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}
\]
2) Circle all of the answers that satisfy the questions below. It is possible that any number of the answers (including none) satisfy the questions. (Complete solutions receive 2 points, partial solutions 1 points, but any incorrect circled answer leads to 0 points.)

a) Let $A$ be an $m \times n$ matrix. Which of the following is equal to $m$? Solution: $v$.
   
   i) $\text{rank}(A)$
   ii) $\dim \text{Col}(A) + \dim \text{Nul}(A)$
   iii) $\text{rank}(A^T)$
   iv) $\dim \text{Col}(A^T) - \dim \text{Nul}(A^T)$
   v) $\dim \text{Col}(A^T) + \dim \text{Nul}(A^T)$

b) Which of the following matrices is in reduced row echelon form? Solution: $iii, v$.
   
   i) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$
   ii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 \end{bmatrix}$
   iii) $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
   iv) $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
   v) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) Which of the following conditions insures an $m \times n$ matrix $A$ is invertible? Solution: $iv, v$.
   
   i) $m = n$.
   ii) There exists an $n \times m$ matrix $B$ such that $AB = I_m$.
   iii) The row echelon form of $A$ has the same number of pivot rows as pivot columns.
   iv) $Ax = b$ has a unique solution $x$ for every $b$.
   v) $A$ is injective and surjective.

d) Which of the following $T : \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation? Solution: $ii, iv, v$.
   
   i) $T(x, y) = x + y + 1$
   ii) $T(x, y) = x - 2y$
   iii) $T(x, y) = x^2 + y^2 - (x + y)^2$
   iv) $T(x, y) = 6(x + 1) + 2(y - 3)$
   v) $T(x, y) = 0$

e) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ has 2-dimensional range and we know
   
   $T(e_1) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
   $T(e_3) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

   Which of the following is a possible value of $T(e_2)$? Solution: $i, iii, iv, v$.
   
   i) $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$
   ii) $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$
   iii) $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$
   iv) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
   v) $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$
3) Consider the matrix

a) (5 points) Find bases for the column space and null space of

\[ A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & -1 & -1 & 1
\end{bmatrix} \]

Row reduce to find:

\[ \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \]

**Null(A) basis:**

\[ \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
1 \\
0
\end{bmatrix} \]

**Col(A) basis:**

\[ \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix} \]

b) (5 points) For what values of \( c \) is the vector

\[ \mathbf{v} = \begin{bmatrix}
c \\
2c \\
c^2
\end{bmatrix} \]

in the column space of \( A \)?

Solve system:

\[ a \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} + b \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix} = \begin{bmatrix}
c \\
2c \\
c^2
\end{bmatrix} \]

Can do by row reduction, or observe that must have \( b = c \), \( a = 2c \) and so \( 2c - c = c^2 \). Thus need \( c = 0 \) or 1.
4) (10 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies the following:

$$T\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T\left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find the standard matrix of $T$.

We seek the $2 \times 3$ matrix with columns $T(e_1), T(e_2)$.

We have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1/2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1/2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1/2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1/2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus we have

$$T(e_1) = T((1/2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1/2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}) = (1/2)T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) + (1/2)T(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$$
$$= (1/2)T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1/2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$T(e_2) = T((1/2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1/2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}) = (1/2)T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) + (-1/2)T(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$$
$$= (1/2)T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1/2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

And so the matrix we seek is

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 0 & 0 \end{bmatrix}$$
5) Decide if each of the following matrices is invertible, and either find its inverse or justify why it is not invertible.

a) (5 points)

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Row reduce to find inverse:

\[
\begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

b) (5 points)

\[ B = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -4 & 4 & 2 & 2 \\ -2 & -4 & -4 & -2 \\ 1 & 2 & -2 & 1 \end{bmatrix} \]

Not invertible: column 2 is twice column 4 so \( \det B = 0 \).
6) (10 points) Suppose that $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$ and that $A$ is an $m \times n$ matrix. Prove that if $Av_1, \ldots, Av_k$ are linearly independent in $\mathbb{R}^m$, then $v_1, \ldots, v_k$ are linearly independent.

Suppose $a_1v_1 + \cdots + a_kv_k = 0$. We must show $a_1 = \cdots = a_k = 0$.

Apply $A$ to find $A(a_1v_1 + \cdots + a_kv_k) = 0$, and so $A(a_1v_1) + \cdots + A(a_kv_k) = 0$, and so $a_1Av_1 + \cdots + a_kAv_k = 0$.

Since $Av_1, \ldots, Av_k$ are linearly independent, we have $a_1 = \cdots = a_k = 0$. 
