This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points, of which you must complete 5. **Choose one problem not to be graded by crossing it out in the box below.** If you forget to cross out a problem, we will roll a die to choose one for you.

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Problem 1) Decide if the following statements are ALWAYS TRUE or SOMETIMES FALSE. You do not need to justify your answers. Enter your answers of T or F in the boxes of the chart. (Correct answers receive 2 points, incorrect answers -2 points, blank answers 0 points.)

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<td>Answer</td>
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1) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by a matrix $A$, then the range of $T$ is equal to the column space of $A$. Both subspaces are the span of the columns of $A$.

2) If two matrices have equal reduced row echelon forms, then their column spaces are equal. Counterexample:

$$
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
$$

3) If a finite set of vectors spans a vector space, then some subset of the vectors is a basis.

If the set is linearly independent, then it is a basis. If not, some vector is a linear combination of the others. Throw out that vector and check that remaining set still spans. Repeat until set is linearly independent.

4) If $A$ is a $2 \times 2$ matrix such that $A^2 = 0$, then $A = 0$.

Counterexample:

$$
A = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
$$

5) If $A$ is a $5 \times 5$ matrix such that $\det(2A) = 2 \det(A)$, then $A = 0$.

Counterexample:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$
Problem 2) Indicate with an X in the chart all of the answers that satisfy the questions below. You do not need to justify your answers. It is possible that any number of the answers (including possibly none) satisfy the questions. (A completely correct row of the chart receives 2 points, a partially correct row receives 1 point, but any incorrect X in a row leads to 0 points.)

<table>
<thead>
<tr>
<th>Question</th>
<th>(a)</th>
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<td>Question 2</td>
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<td>Question 3</td>
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<td>Question 4</td>
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<td>Question 5</td>
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<td>X</td>
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</table>

Inside of \( \mathbb{R}^3 \), consider the vectors

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{v}_2 &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\
\mathbf{v}_3 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{v}_4 &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
\mathbf{v}_5 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
\mathbf{v}_6 &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
\end{align*}
\]

1) Which of the following lists are linearly independent?
   a) \( \mathbf{v}_1, \mathbf{v}_2 \).
   b) \( \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \).
   c) \( \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5 \).
   d) \( \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6 \).
   e) \( \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \).

2) Which of the following lists span \( \mathbb{R}^3 \)?
   a) \( \mathbf{v}_1, \mathbf{v}_2 \).
   b) \( \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \).
   c) \( \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5 \).
   d) \( \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6 \).
   e) \( \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \).
3) Which of the following matrices have reduced row echelon form equal to \[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]?

a) \[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]
b) \[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
2 & 2 & 2 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
d) \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]
e) \[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

4) Inside of \(\mathbb{R}^3\), consider the subset of vectors 
\[
\{ v = \begin{bmatrix} a \\ b \\ a \end{bmatrix} \}
\]
satisfying the following requirements. Which of them are subspaces?

a) \(a\) and \(b\) are both zero.
b) \(a\) is any number and \(b\) is zero.
c) \(a\) is zero or \(b\) is zero or both are zero.
d) \(a\) and \(b\) are equal.
e) \(a, b\) are both positive, both negative, or both zero.

5) Suppose \(T : \mathbb{R}^3 \to \mathbb{R}^3\) has 2-dimensional range and we know 
\[
T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
Which of the following are a possible standard matrix of \(T\)?

a) \[
\begin{bmatrix}
-1 & 1 & 0 \\
-2 & 2 & 0 \\
1 & -1 & 0
\end{bmatrix}
\]
b) \[
\begin{bmatrix}
1 & 1 & -2 \\
2 & 2 & -4 \\
1 & -1 & -2
\end{bmatrix}
\]
c) \[
\begin{bmatrix}
1 & 1 & -2 \\
0 & 2 & -2 \\
0 & -1 & 1
\end{bmatrix}
\]
d) \[
\begin{bmatrix}
1 & 1 & -2 \\
0 & 2 & -2 \\
2 & -1 & -1
\end{bmatrix}
\]
e) \[
\begin{bmatrix}
-1 & 1 & 0 \\
0 & 2 & -2 \\
0 & -1 & 1
\end{bmatrix}
\]
Problem 3) For a real number $c$, consider the linear system
\[
\begin{align*}
 x_1 &+ x_2 + cx_3 + x_4 = c \\
 -x_2 + x_3 + 2x_4 &= 0 \\
 x_1 + 2x_2 + x_3 - x_4 &= -c
\end{align*}
\]

a) (5 points) For what $c$, does the linear system have a solution?

Let us find the REF of the augmented matrix
\[
\begin{bmatrix}
1 & 1 & c & 1 & | & c \\
0 & -1 & 1 & 2 & | & 0 \\
1 & 2 & 1 & -1 & | & -c
\end{bmatrix} \Rightarrow 
\begin{bmatrix}
1 & 1 & c & 1 & | & c \\
0 & -1 & 1 & 2 & | & 0 \\
0 & 1 & 1 - c & -2 & | & -2c
\end{bmatrix} \Rightarrow 
\begin{bmatrix}
1 & 1 & c & 1 & | & c \\
0 & -1 & 1 & 2 & | & 0 \\
0 & 0 & 2 - c & 0 & | & -2c
\end{bmatrix}
\]

Thus the linear system has a solution if and only if $c \neq 2$.

b) (5 points) Find a basis of the subspace of solutions when $c = 0$.

When $c = 0$, the REF of the unaugmented matrix is
\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & -1 & 1 & 2 \\
0 & 0 & 2 & 0
\end{bmatrix}
\]

The free variable is $x_4$ and so solutions are of the form
\[
\begin{bmatrix}
-3x_4 \\
2x_4 \\
x_4
\end{bmatrix}
\]

Thus a basis consists of the single vector
\[
\begin{bmatrix}
-3 \\
2 \\
0 \\
1
\end{bmatrix}
\]
Problem 4) (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree less than or equal to 2. Let $B$ be the basis $b_1 = x^2, b_2 = -1 + x, b_3 = x + x^2$.

Find the coordinates of the vector $v = 1 + 2x - x^2$ with respect to $B$.

Writing all polynomials in terms of the standard basis $1, x, x^2$, we find we must solve the linear system with augmented matrix

$$
\begin{bmatrix}
0 & -1 & 0 & 1 \\
0 & 1 & 1 & 2 \\
1 & 0 & 1 & -1 \\
\end{bmatrix}
$$

Let us put it into REF

$$
\begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 1 & 2 \\
0 & -1 & 0 & 1 \\
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
$$

Now we find the solution which is the sought-after coordinate vector

$$
[v]_B = \begin{bmatrix}
-4 \\
-1 \\
3 \\
\end{bmatrix}
$$
Problem 5) Consider the matrices

\[ A = \begin{bmatrix}
1 & 0 & -1 \\
2 & 1 & 0 \\
0 & 1 & 0 \\
-1 & 1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 1 & 0 \\
2 & 1 & 0 & -1 \\
0 & -1 & 2 & -1
\end{bmatrix} \]

1) (5 points) Calculate the matrix \( AB \).

\[ AB = \begin{bmatrix}
1 & 1 & -1 & 1 \\
4 & 1 & 2 & -1 \\
2 & 1 & 0 & -1 \\
1 & -1 & 3 & -3
\end{bmatrix} \]

2) (5 points) Calculate the determinant \( \det(AB) \). Cite any methods used in your answer.

\[ \det(AB) = 0 \] since \( AB \) is not invertible. The reason \( AB \) is not invertible is \( AB \) has a nontrivial null space. The reason \( AB \) has a nontrivial null space is that \( B \) maps \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \) and so \( B \) must have a nontrivial null space, and \( AB \) results from first applying \( B \) then \( A \), so any vector in the null space of \( B \) will be in the null space of \( AB \).
Problem 6)

1) (6 points) Fill in the blanks (each worth 1/2 a point) in the proof of the following assertion.

Assertion. If $A$ is a square matrix, and the linear transformation $x \mapsto Ax$ is injective, then the linear transformation $x \mapsto A^T x$ is injective.

Proof. For any $m \times n$ matrix $A$, recall that

$$n = \text{rank}(A) + \text{dim} \text{Nul}(A)$$

and similarly for $A^T$, we have

$$m = \text{rank}(A^T) + \text{dim} \text{Nul}(A^T)$$

We also know for $A$ and $A^T$ that

$$\text{rank}(A) = \text{rank}(A^T)$$

Next recall that $x \mapsto Ax$ is injective if and only if

$$\text{dim} \text{Nul}(A) = 0$$

and similarly, $x \mapsto A^T x$ is injective if and only if

$$\text{dim} \text{Nul}(A^T) = 0$$

Thus when $A$ is square, so $m = n$, and $x \mapsto Ax$ is injective, we have

$$\text{rank}(A) = n = m = \text{rank}(A^T) + \text{dim} \text{Nul}(A^T)$$

And so we conclude that

$$\text{dim} \text{Nul}(A^T) = 0$$

and hence $x \mapsto A^T x$ is injective.

2) (4 points) Give an example of a $2 \times 2$ matrix $A$ such that $\text{Nul}(A) \neq \text{Nul}(A^T)$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{Nul}(A) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$$

$$A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{Nul}(A^T) = \text{Span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$