Congratulations on Surviving
Determinants!

And Welcome to Lecture 9.

Today: Office Hours 12:30-2:30
736 Evans

Fri Quiz through 3.3

__Warmup 1__ For what a is A invt-ble?

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 0 & -1 & 2 \\ 2a & 2a & 2a \end{bmatrix}$$

__Soln__ $\det A = 2a \det \begin{bmatrix} 0 & 0 & -3 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$= -2a \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= (-2a) \cdot 1 \cdot (-1) \cdot (-3) = -6a$$

Invt-ble $\Updownarrow$ $a \neq 0$

Warmup 2 For what a is A int-ble?

$$A = \left[\begin{array}{cc|cc} 2 & -1 & 0 & 0 \\ a & 2 & 0 & 0 \\ \hline a & a^2 & 2 & 1 \\ -a^2 & a & -a & 2 \end{array}\right]$$

$$\rightsquigarrow \begin{bmatrix} 2+\frac{a}{2} & 0 & 0 & 0 \\ a & 2 & 0 & 0 \\ a+\frac{a^2}{2} & a^2-\frac{a}{2} & 2+\frac{a}{2} & 0 \\ -a^2 & a & -a & 2 \end{bmatrix}$$

So $\det A = (2+\frac{a}{2})\cdot 2\cdot(2+\frac{a}{2})\cdot 2$

A invt-ble $\iff a \neq -4$.

<u>Aside</u>: Trick formulas for 2×2, 3×3 dets.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

(In fact this is the def!)

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + dhc - ceg - bdi - afh$$

__Warmup__ 3 Find basis in terms of a for Col A where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & a & -2 \end{bmatrix}$$

__Strategy__ Put in REF

Find pivot cols

Take ξ corresponding cols of A (__not__ cols of REF!)

$$\rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & a & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & a & 0 \end{pmatrix}$$

Two cases:

$\underline{a=0}$ $\quad \begin{bmatrix} \boxed{1} & -1 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$ REF $\quad$ Basis $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$\underline{a \neq 0}$ $\quad \begin{bmatrix} \boxed{1} & -1 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{a}_{\neq 0} \end{bmatrix}$ REF $\quad$ Basis $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ a \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

State list of all props of $\mathbb{R}^n$ we ever use:

1) There is addition of vectors:

$$\underline{u}, \underline{v} \rightsquigarrow \underline{u} + \underline{v}$$

2) $+$ is commutative: $\underline{u} + \underline{v} = \underline{v} + \underline{u}$

3) $+$ is associative: $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

4) There is a zero vector $\underline{0}$ such that

$$\underline{u} + \underline{0} = \underline{u}$$

5) There is a negative vector $\underline{u} \rightsquigarrow -\underline{u}$

such that $\underline{u} + (-\underline{u}) = \underline{0}$

6) There is scaling of vectors by numbers

$c, \underline{u} \rightsquigarrow c \cdot \underline{u}$

7) Distribuitive in scalar:

$(c+d) \cdot \underline{u} = c\underline{u} + d \cdot \underline{u}$

8) Dist. in vector:

$c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$

9) Assoc.: $c(d\underline{u}) = (cd) \cdot \underline{u}$

10) $1 \cdot \underline{u} = \underline{u}$

Def A vector space V is a set (whose elements we call vectors) with two operations: addition & scaling by numbers

Such that properties 1) - 10) hold.

Ex 1) $V = \mathbb{R}^n$

2) $V = \{ \text{functions } f : \mathbb{R} \to \mathbb{R} \}$

$$(f+g)(x) = f(x) + g(x)$$

$$(cf)(x) = cf(x)$$

3) $V = \mathbb{P} = \{ \text{polynomial functions } f : \mathbb{R} \to \mathbb{R} \}$

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

for some n.

Same add, scale as on all fns.

4) $V = \mathbb{P}_n = \{$ poly fns $f : \mathbb{R} \to \mathbb{R}$ of deg $\leq n \}$

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$

a_i's are possibly 0

$\triangle$ 2 poly fns of deg $= n$ is <u>not</u> a vector space!

5) $V = \mathbb{S} = \{$ sequences of numbers $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \}$

<u>Meta-example</u>: Real-valued fns on <u>anything</u> is an example of a vector space

<u>Example</u> Temp on Earth's surface

...

Def. Let V be a vector space. A _subspace_ H of V is a subset of vectors such that:

1) $\underline{0}$ is in H
2) If $\underline{u}, \underline{v}$ are in H, then $\underline{u}+\underline{v}$ is also in H.
3) If $\underline{u}$ is in H, then $c\underline{u}$ is also in H for any c.

Ex 1) $H = \{0\}$ or $H = V$ itself

2) $H = \mathbb{P}_n$ inside $V = \mathbb{P}$

3) A $m \times n$ matrix

Nul A is a subspace of $\mathbb{R}^n$

Col A is a subspace of $\mathbb{R}^m$

1) Nul A, 2) Col A exemplify <u>two</u> ways subspaces arise

1) Subspace H is cut out by eqns.
   Challenging to find vectors in H.
   Easy to check if a vector is in H.

2) Subspace H is spanned by some vectors.
   Easy to find vectors in H.
   Challenging to check if a vector is in H.

__Def.__ Let V, W be vector spaces.
A __linear transformation__ $T: V \to W$
is a map such that

1) $T(\underline{u} + \underline{v}) = T\underline{u} + T\underline{v}$
2) $T(c\underline{u}) = cT\underline{u}$

Ex 1) $V=\mathbb{R}^n$, $W=\mathbb{R}^m$, T must be given by a $m\times n$ matrix A.

2) $V=W=\mathbb{P}$ polynomials

a) $T=$ mult. by x

$$T(p(x)) = xp(x)$$

If $p(x)=a_0+a_1x+\cdots+a_nx^n$

then $T(p(x))=a_0x+a_1x^2+\cdots+a_nx^{n+1}$

b) $T =$ derivative $\frac{d}{dx}$

$$T(p(x)) = \frac{dp}{dx}(x)$$

If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$$T(p(x)) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

c) $T =$ act by $x\frac{d}{dx}$

$$T(p(x)) = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n$$

34) $V = \mathbb{P}_n$, $W = \mathbb{R}^2$, $T : V \to W$

$$T(p(x)) = \begin{bmatrix} p(2) \\ \frac{dp}{dx}(-1) \end{bmatrix}$$

Check this is a lin transf!

~~Exerc~~

<u>Def</u> $T: V \to W$ lin. transf.

$$\text{Nul}\,T = \{\, \underline{v} \text{ in } V \text{ s.t. } T\underline{v} = \underline{0} \,\}$$

$$\text{Range}\,T = \{\, \underline{w} \text{ in } W \text{ s.t. there is } \underline{v} \text{ in } V \text{ with } T\underline{v} = \underline{w} \,\}$$

or

Image T ← Same as Col A for T given by A.

Exer What are Nul T and Range T for $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ given by

$$T(p(x)) = \frac{dp}{dx}(x)$$

$$T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$$

$$\text{Nul } T = \{ \ p(x) = a_0 \quad \text{const polys} \}$$

$$\text{Range } T = \{ \ p(x) = b_0 + b_1 x \quad \text{polys of deg} \le 1 \}$$