Welcome to Lecture 7

Please be Kind to guest lecturer

Today: No office hours
→ See course webpage for GSI office hours

Friday: Quiz up to and including 2.6

BUMP - Berkeley Undergraduate Mentoring Program

Info session: TODAY @ 6pm
1015 EVANS

Warmup Problem:

Recall: A **basis** for a subspace H of $\mathbb{R}^n$ is a list of vectors $\underline{v}_1, \ldots, \underline{v}_k$ in H that **both**

a) Span H

b) Linearly independent

Goldilocks + 3 Bears.

Back to warmup:

Find a basis for the column space Col A and the null space Nul A of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

<u>Solution</u>: Put A into REF

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to 2R_1 - R_2]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Basis for Column space Col A.

Pivot columns are columns 1 + 2

A basis for Col A is a list of corresponding pivot columns from A

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>Warning</u>: Do not use REF matrix columns.

Basis for null space Nul A:

Pivot columns are 1 + 2

x_1 and x_2 are pivot variables and x_3 is a free variable

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = -x_2 \qquad x_2 = x_3$

$$\underline{x} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Nul A is spanned by $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\underline{v}_1$ is a basis for Nul A.

Col A had 2 vectors in basis

Nul A had 1 vecto in basis

$2 + 1 = 3 =$ # columns of A

<u>Warmup 2</u>: Find a basis for Col A and Nul A for

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Already in REF</u>!

Pivot in first column, Hence a basis for Col A is the first column of A

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Pivot variable is x_1

Free variables are x_2, x_3, x_4

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 0 \end{bmatrix}$$

$$x_1 = -2x_2 - 3x_3 - 4x_4$$

Any vector $\underline{x}$ in Nul A, is of the form

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 - 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\underline{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a basis for Nul A is

$$\underline{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

vectors in basis of Col A = 1

vectors in basis of Nul A = 3

$1 + 3 = 4 =$ # Columns of A

Key Observation for today:

$n =$ # columns of A = # pivot columns + # free columns

= # pivot variables + # free variables

= Size of basis for Col A + Size of basis for Nul A

What is a basis good for?

Let $\underline{b}_1, \dots, \underline{b}_k$ be a basis for a subspace H of $\mathbb{R}^n$.

Basis allows us to express $\underline{x}$ in H in coordinates relative to $\underline{b}_1, \dots, \underline{b}_k$.

Since $\underline{b}_1, \dots, \underline{b}_k$ spans H, there exists $c_1, \dots, c_k$ such that

$$\underline{x} = c_1 \underline{b}_1 + \dots + c_k \underline{b}_k .$$

$c_1, \dots, c_k$ are unique!

Suppose that for $\underline{x}$ in H, we have

$$\underline{x} = c_1\underline{b}_1 + \cdots + c_k\underline{b}_k = c_1'\underline{b}_1 + \cdots + c_k'\underline{b}_k$$

$$0 = (\underbrace{c_1 - c_1'}_{=0})\underline{b}_1 + \cdots + (\underbrace{c_k - c_k'}_{=0})\underline{b}_k$$

Since $\underline{b}_1, \ldots, \underline{b}_k$ are $\boxed{\text{linearly independent}}$

we have $c_1 - c_1' = c_2 - c_2' = \cdots = c_k - c_k' = 0$

Hence $c_1 = c_1'$, $c_2 = c_2'$, $\ldots$, $c_k = c_k'$

let us write $\mathcal{B}$ for a basis $\underline{b}_1, \dots, \underline{b}_k$

<u>Def</u>: For any $\underline{x}$ in H, the $\boxed{\text{coordinates}}$ of $\underline{x}$ with respect to the basis $\mathcal{B}$ are the numbers $c_1, \dots, c_k$ such that

$$\underline{x} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + \cdots + c_k \underline{b}_k$$

We organize $c_1, \dots, c_k$ into a vector called the coordinate vector

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Define H in $\mathbb{R}^4$ to be the subspace spanned by the basis

$$\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

what are the coordinates of

$$\underline{x} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \; ?$$

Need to find c_1, c_2, c_3 so that

$$\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3$$

$$\underline{x} = \begin{bmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Equivalent to solving the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 0 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Put into REF

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 0 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right] \underset{R_2 \to R_1 + R_2}{\leadsto} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_2 + R_3}{\leadsto} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\underset{R_4 \leadsto R_3 + R_4}{\leadsto} \text{REF} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 = 0 \qquad c_2 = 1 \qquad c_1 + c_2 = 3$

$$c_1 + 1 = 3$$

$$c_1 = 2$$

Solution is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

The coordinates of $\underline{x}$ in $\mathcal{B}$ are

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Recap: $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ means that

$$\underline{x} = 2\underline{v}_1 + 1 \cdot \underline{v}_2 + 0 \cdot \underline{v}_3$$

$$\underline{x} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = 2 \underset{\substack{\uparrow \\ \underline{v}_1}}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} + 1 \cdot \underset{\substack{\uparrow \\ \underline{v}_2}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} + 0 \cdot \underset{\substack{\uparrow \\ \underline{v}_3}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}$$

$$A = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix}$$

$$\underline{x} = A \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2\underline{v}_1 + \underline{v}_2 + 0 \cdot \underline{v}_3$$

<u>Caution:</u> <u>Coordinates</u> are completely different numbers from original components of $\underline{x}$.

It turns out that any two bases for a subspace H of $\mathbb{R}^n$ must have the same number of vectors.

<u>Def</u>: The <u>dimension</u> of a subspace H is the number of vectors in any basis for H.

We define the dimension of $\{0\}$ to be 0.

<u>Ex</u>: The dimension of $\mathbb{R}^n$ is n.

(Very Important) Rank theorem

For any ~~[illegible]~~ $m \times n$ matrix A

$$n = \underbrace{\text{dimension}(\text{Col} A)}_{\text{rank}(A)} + \dim(\text{Nul} A)$$

<u>Def</u>: The <u>rank</u> of a matrix A is the dimension of Col A.

(very important theorem) **Basis theorem**

If H is a k-dimensional subspace then $\boxed{\text{any}}$ k linearly independent vectors $\boxed{\text{span}}$ H. Also, any k vectors that span H are linearly independent.

We can reformulate what it means for A to be invertible:

<u>Theorem</u>: An $n \times n$ matrix A is invertible if and only if any one of the following conditions hold

1) $\mathrm{Col}\, A = \mathbb{R}^n$

1') $\dim(\mathrm{Col}\, A) = \mathrm{rank}(A) = n$

2) $\mathrm{Nul}(A) = \{0\}$

2') $\dim(\mathrm{Nul}\, A) = 0$

1), 1') A is surjective (onto)   2) 2') A is injective (1-1).

Ex: Find dimensions of Col A and Nul A as we vary c, where

$$A = \begin{bmatrix} c^2 & 2c-1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= c^2 \cdot 1 - (2c-1) \cdot 1 \\ &= c^2 - 2c + 1 \\ &= (c-1)^2 \end{aligned}$$

A is invertible when $\det(A) \neq 0$

when $\boxed{c \neq 1}$

When $c \neq 1$, A is invertible, and

$\text{rank}(A) = \dim(\text{Col}\,A) = 2$

$\dim(\text{Nul}\,A) = 0$

When $c = 1$: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\underset{R_2 \to R_1 - R_2}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

1 pivot column

1 free column

$\text{rank}(A) = \dim(\text{Col}\,A) = 1$

$\dim(\text{Nul}\,A) = 1$