Welcome to Lecture 6!

This week:

Thurs: Guest Lecturer

No Office Hours ☹

Fri: Quiz through 2.6

<u>Warmup</u> Are the following matrices invt-ble and if so find inverses.

1) $A = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$

determinant $= (2)(-1) - (1)(-3) = 1 \neq 0$

$A^{-1} = \frac{1}{1}\begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix}$

2) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ ✗

No chance to be invt-ble

Not square.

3) $A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

Algorithm to find answer:

$$\left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cc} A & I_3 \end{array}$$

Try to put A in RREF working with entire extended matrix

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

row swap 1 & 2

$E_1 A$ $\quad$ $E_1 I_3$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

subtract row 1 from row 2

$\boxed{E_2 E_1 A}$ $\quad$ $\boxed{E_2 E_1 I_3}$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$\boxed{E_3E_2E_1A} \qquad \boxed{E_3E_2E_1I_3}$$

scale row 3 by $\frac{1}{2}$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$\underbrace{\boxed{E_4E_3E_2E_1A}}_{I_3} \qquad \overbrace{\boxed{E_4E_3E_2E_1I_3}}^{A^{-1}}$$

add row 3 to row 2

If you discover there are less than n pivots, then conclusion:

not invertible!

not a pivot in each row & col.

Why does this algorithm work?

Row swap of rows 1 & 2 given by left multiplying A by the matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtracting row 1 from row 2 given by left mult. by the matrix

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding row 3 to row 2 given by left mult by the matrix

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Conclusion</u> $E_4E_3E_2E_1A = I_3$

So $A^{-1} = E_4E_3E_2E_1$!

<u>Note</u>: each E_i is invt-ble!

So $\cancel{E_1^{-1}}\cancel{E_2^{-1}}\cancel{E_3^{-1}}\cancel{E_4^{-1}}\cancel{E_4}\cancel{E_3}\cancel{E_2}\cancel{E_1}A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}I_3$

So $AE_4E_3E_2E_1 = \cancel{E_1^{-1}}\cancel{E_2^{-1}}\cancel{E_3^{-1}}\cancel{E_4^{-1}}\cancel{E_4}\cancel{E_3}\cancel{E_2}\cancel{E_1}$

$= I_3$

Theorem For A an $n \times n$ matrix, the following are equivalent:

1) A is invertible

2) RREF of A is I_n

2') REF of A has pivot in each col

2'') REF of A has pivot in each row

3) $A\underline{x} = \underline{0}$ has only triv soln $\underline{x} = \underline{0}$

3') Cols of A are lin indep

3'') Lin transf given by A is inj.

4) $A\underline{x} = \underline{b}$ has soln for all $\underline{b}$

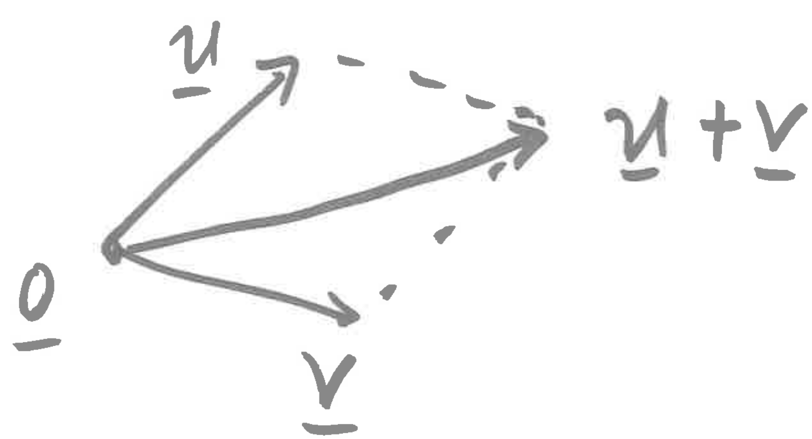
4') Cols of A span $\mathbb{R}^n$

4'') Lin transf given by A is surj.

Subspaces

<u>Def.</u> A <u>subspace</u> H of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ satisfying:

1) $\underline{0}$ is in H
2) if $\underline{u}, \underline{v}$ in H, then $\underline{u}+\underline{v}$ in H



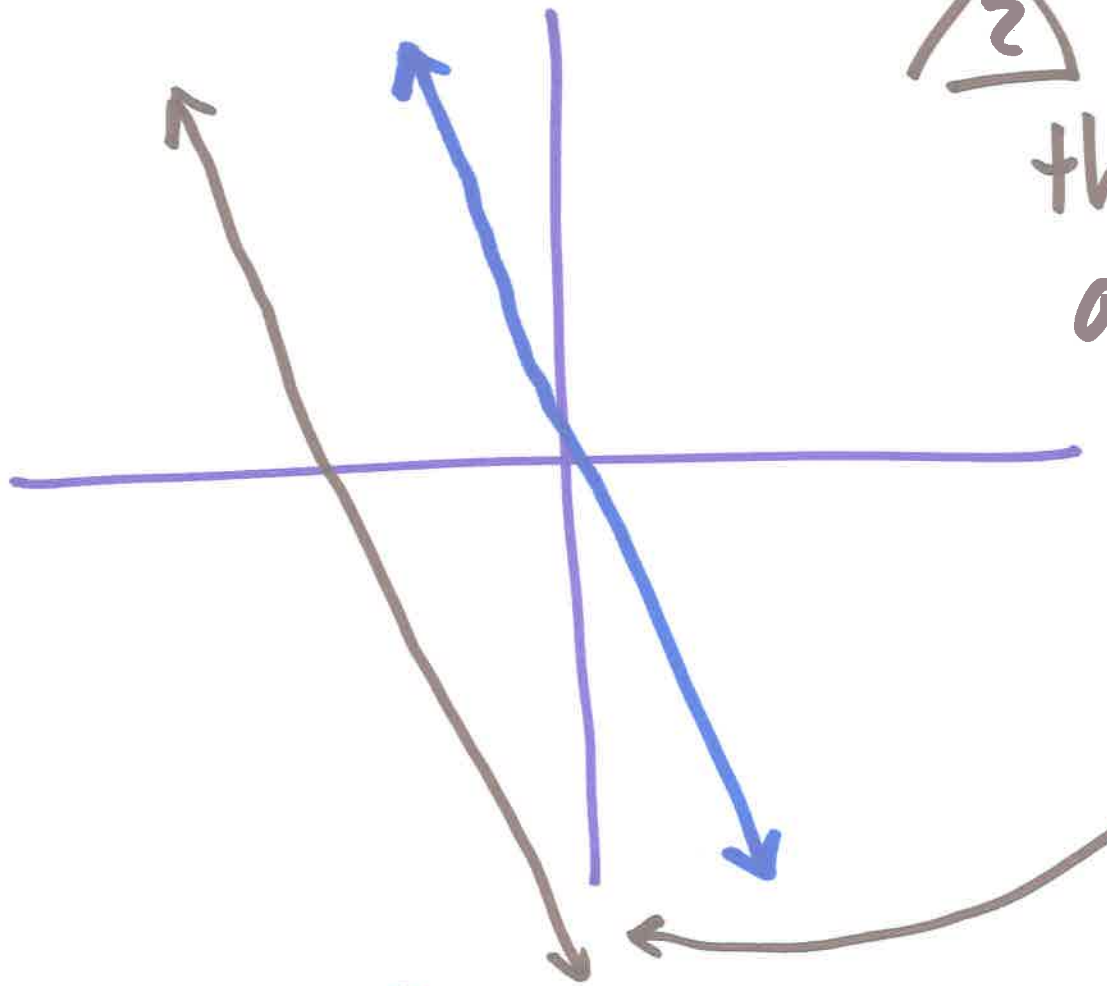
3) if $\underline{u}$ in H, then $c\underline{u}$ in H for all c.

<u>Examples</u>

1) <u>$n=1$</u> Subspaces of $\mathbb{R}$

<u>Possibilities</u>: $\mathbb{R}$ itself, $\{\underline{0}\}$ zero subspace.

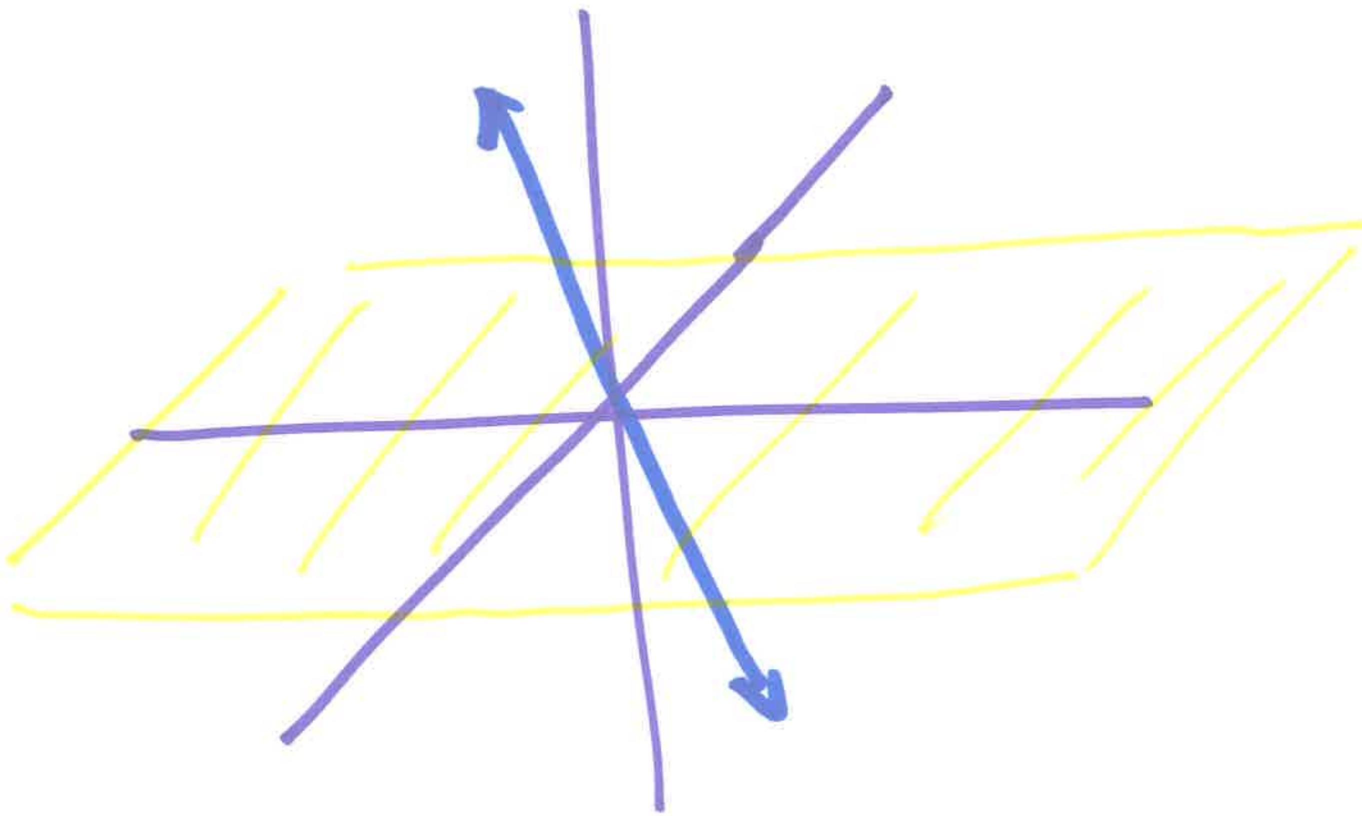
2) __$n=2$__ subspaces of $\mathbb{R}^2$



⚠ lines __not__ through $\underline{0}$ are __not__ subspaces

__Possibilities__: $\mathbb{R}^2$, $\{\underline{0}\}$, line through $\underline{0}$

3) $\underline{n=3}$ subspaces of $\mathbb{R}^3$



<u>Possibilities</u>: $\mathbb{R}^3$, $\{\underline{0}\}$, lines through $\underline{0}$, planes through $\underline{0}$

Theorem Let $\underline{v}_1, \ldots, \underline{v}_k$ list of vectors in $\mathbb{R}^n$. Then the span of $\underline{v}_1, \ldots, \underline{v}_k$ is a subspace.

$$\text{Span} = \{ a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k \}$$

any numbers.

Proof: Check for yourself!

<u>Two most important examples</u>

Let A be an $m \times n$ matrix

$$m\left\{\underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{n}\right.$$

<u>Def</u> 1) <u>Column space</u> of A is the span of the cols of A

(= Range / Image of A)

2) <u>Null space</u> of A is the

space of solns of $A\underline{x} = \underline{0}$

(= Soln set of $A\underline{x} = \underline{0}$)

<u>Theorem</u> Col space & Null space are subspaces.

<u>Proof</u> Check for yourself!

(Very important) Def. A **basis** for a subspace H is a list of vectors $\underline{v}_1, \dots, \underline{v}_k$ in H that is **lin indep** and **spans** H.

$a_1\underline{v}_1 + \dots + a_k\underline{v}_k = \underline{0}$

$\Rightarrow a_1 = \dots = a_k = 0$

Every $\underline{v}$ in H is a lin comb

$\underline{v} = a_1\underline{v}_1 + \dots + a_k\underline{v}_k$

Silly (but extremely important to not be confused about) example:

Take $H = \mathbb{R}^n$ itself

Coord. vectors form a basis

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Exer Find bases for col and null spaces of $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

pivot cols of original matrix

Soln: Put A in REF

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

pivot cols

Basis for col space: take pivot cols of original matrix

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Basic for null space: to be continued...