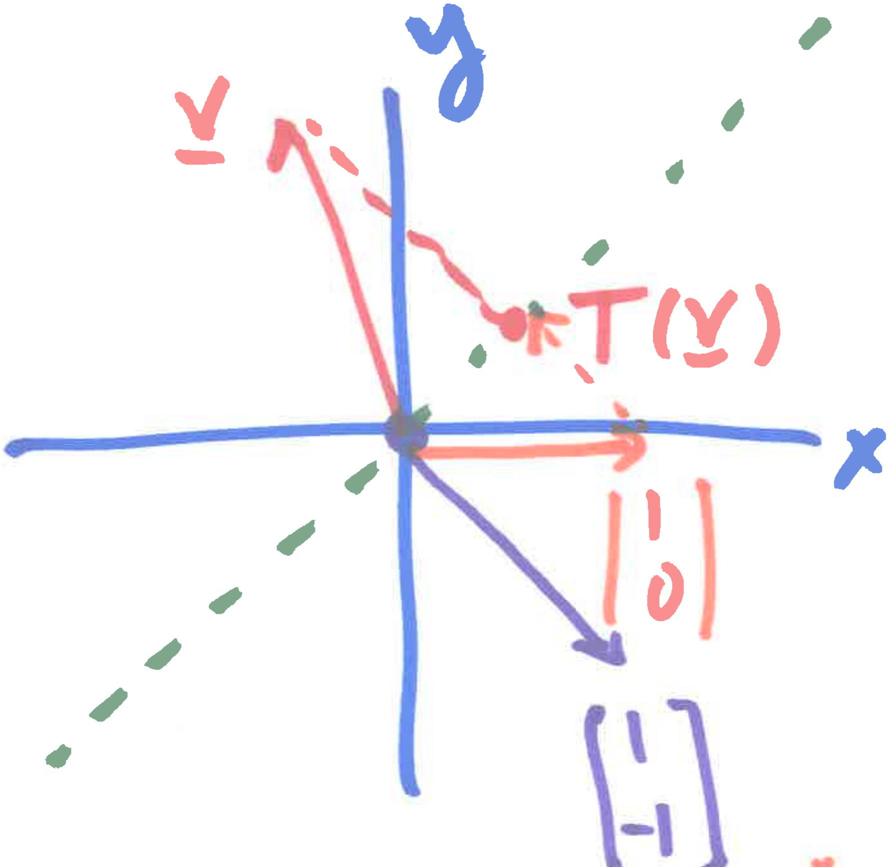
Welcome to Lecture 5!

Today: Office Hours 12:30-2:30
736 Evans

Friday Quiz up to and including
section 1.9

Warmup! Find matrix for lin transf $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that projects to $x=y$.



$$A = \begin{bmatrix} | & | \\ T(\underline{e}_1) & T(\underline{e}_2) \\ | & | \end{bmatrix}$$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Is T injective / one-to-one?

<u>No</u>! $T(\underline{0}) = \underline{0}$ $\quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \underline{0}$

Is T surjective / onto?

<u>No</u> For example, T never takes value $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2) Is $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $A = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 0 & -1 \end{bmatrix}$ injective?

<u>Recall</u>: $T: \mathbb{R}^n \to \mathbb{R}^m$ given by matrix A is injective $\iff$ cols of A lin indep

($\iff$ pivot in each col of RREF

$\iff$ $\boxed{A\underline{x} = \underline{0} \text{ has only triv soln}}$

Why? T injective means: $T\underline{x} = T\underline{y}$
then $\underline{x} = \underline{y}$

Suppose $A\underline{x} = \underline{0}$. We know $A\underline{0} = \underline{0}$

So $\underline{x} = \underline{0}$ since T inj.

Suppose $A\underline{x} = 0$ has only triv soln.

But suppose $A\underline{v} = A\underline{w}$ So $\underline{v} = \underline{w}$.

Then $A\underline{v} - A\underline{w} = \underline{0}$

So $A(\underline{v} - \underline{w}) = \underline{0}$ So $\underline{v} - \underline{w} = \underline{0}$

Back to problem:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Yes inj! pivot in each col.

Is T surj? Impossible since

$$\underbrace{2}_{\text{domain dim}} < \underbrace{3}_{\text{codomain dim}}$$

3) For what c, ~~Is~~ is $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$A = \begin{vmatrix} 0 & 1 \\ c & 0 \end{vmatrix}$ inj? surj?

$c = 0 \rightsquigarrow \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$ RREF No, No

$c \neq 0 \rightsquigarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ RREF Yes, Yes

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{array}\right]$$

no soln!

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right]$$

exists soln and <u>unique</u>!

Matrix algebra

What can we do with matrices?

add: A, B $m \times n$ matrices

then $A+B$ is $m \times n$ matrix

with entries $(A+B)_{ij} = A_{ij} + B_{ij}$

scale A $m \times n$ matrix, c number

then cA is $m \times n$ matrix

with entries $(cA)_{ij} = cA_{ij}$

Special matrices

0 denotes $m \times n$ matrix with all zero entries

$-A$ denotes $m \times n$ matrix with entries $(-A)_{ij} = -(A_{ij})$

$$I_n = \underbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{n} \Big\} n$$

identity matrix

Matrix multiplication

A $m \times n$ matrix, B $n \times p$ matrix

$$m\left\{\underbrace{\begin{bmatrix} \quad & A & \quad \end{bmatrix}}_{n}\right. \quad \underbrace{\begin{bmatrix} \quad \\ \quad & B & \quad \\ \quad \end{bmatrix}}_{p}\Big\} n$$

Then there is a product matrix AB ($m \times p$)

$$AB = \left.\underbrace{\begin{bmatrix} | & | & & | \\ A\underline{b}_1 & A\underline{b}_2 & \cdots & A\underline{b}_p \\ | & | & & | \end{bmatrix}}_{p}\right\} m$$

$$B = \begin{bmatrix} | & | & & | \\ \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_p \\ | & | & & | \end{bmatrix}$$

<u>Example</u>: $A = \begin{bmatrix} 2 & 0 & 3 & 2 \\ 1 & -1 & 4 & 7 \end{bmatrix}$ 2×4

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad 4 \times 3$$

$$AB = \left.\underbrace{\begin{bmatrix} \cancel{2}\ 4 & 2 & -2 \\ 7 & \cancel{1}\ -1 & -1 \end{bmatrix}}_{3}\right\}2$$

$$2 \times 3$$

Many good properties of matrix mult

.... see book.

$\triangle 2$ 1) $AB \neq BA$

not always

Ex. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\neq$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Not <u>commutative</u>!

2) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

not always

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

3) $AB = AC \not\Rightarrow B = C$

$A \neq 0$ not always

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ 0 & -7 \end{pmatrix}$

Cartoon of matrix mult in terms of lin transf.

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \qquad A \;\; m \times n$$

$$\mathbb{R}^p \xrightarrow{B} \mathbb{R}^n \qquad B \;\; n \times p$$

AB = composition of first B then A

$$
\begin{tikzcd}
\mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m
\end{tikzcd}
$$

<u>Exer</u> What does it mean if

$$A \cdot B = 0 \quad ?$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_p \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ A\underline{b}_1 & \cdots & A\underline{b}_p \\ | & & | \end{bmatrix} = 0$$

means $\underline{b}_i$ solve $A\underline{x} = \underline{0}$

Inverses of matrices

Only makes sense for square matrices

Def An $n \times n$ matrix A is invertible with inverse the $n \times n$ matrix A^{-1} if

$$A \cdot A^{-1} = I_n = A^{-1} A$$

__Exer__ Find inverse of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
(May not have inverse...)

__Take__: $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ __Check__: $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$= A^{-1}A$

__Exer__ Show $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ does not have an inverse.

Suppose there were A^{-1} with $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Take $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ apply both sides of

$$A^{-1}A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

↯ contradiction

__Theorem__ $T : \mathbb{R}^n \to \mathbb{R}^n$ lin transf

Same!

(so given by __square__ matrix)

Then T inj $\Longleftrightarrow$ T surj.

__Proof__. Inj $\Longleftrightarrow$ RREF has pivot in each col $\overset{\text{square!}}{\Longleftrightarrow}$ RREF has pivot in each row $\Longleftrightarrow$ Surj

Theorem A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff \boxed{ad-bc \neq 0}$

If so $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$ad-bc$ is called the **determinant** of A

Exer For what c is $A = \begin{bmatrix} 1 & 1 \\ 1 & c \end{bmatrix}$ invertible?

$$1 \cdot c - 1 \cdot 1 = c - 1$$

Invertible $\iff c \neq 1$

If so $A^{-1} = \frac{1}{c-1} \begin{bmatrix} c & -1 \\ -1 & 1 \end{bmatrix}$